VIII All Russian students olympiad on mathematics of higher education (ARSO) in 2022-2023

1. Find limit

$$\lim_{n\to 0^+}\left(\frac{17^n+119^n}{2}\right)^{1/n}$$

2. Find 6 points on the plane that do not lie on a straight line such that the distance between any two of them is an integer.

3. Let $A, B \in M_{n\times n}(\mathbb{C})$, $a, b \in \mathbb{C}/\{0\}$ and $a \neq b$. Find $det(A-B)$, if it is known that

$$AB = aA + bB$$
$$BA = bA + aB.$$

4. Evaluate

$$\int\limits_{|z|=1} \frac{z^{n-1}}{2z^n+1}dz, \quad n \in \mathbb{N}.$$

5. Let x, y be positive integer. Let $\{a_n\}$ be a sequence given by

$$a_1 = 1,\ a_2 = 1+2y,\ a_3 = 1+3x+3y,$$
$$a_{n+3} = xa_n + ya_{n+1} + a_{n+2},\ \text{for } n \geq 1.$$

Prove that for any prime number p, the number $a_p - 1$ is divisible by p.

6. Let f is bounded function on $[0,2]$, satisfies

$$f(t+h) \geq h\left(f(t)^2 + f(t)\right) + 1,$$

with every $t, h \geq 0,\ t+h \leq 2$. Find $f(t)$.

7. Show that exist infinitely many pairs of positive integers (m, n) such that

$$\frac{n}{m} + \frac{m+1}{n}$$

is also positive integer.