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ON SOLVABILITY OF SPATIAL NONLOCAL BOUNDARY VALUE PROBLEMS FOR SOME ANALOGS OF THE BOUSSINESQ EQUATION

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Abstract. Considering an analog of the Boussinesq equation, we examine spatial nonlocal boundary value problems with the Samarskiĭ condition and prove the existence and uniqueness of regular solutions.

Keywords: nonlocal boundary value problem, Boussinesq equation, existence and uniqueness of regular solutions

Assume that Ω is the interval (0,1) of the Ox-axis, Q is the rectangular $\Omega \times (0,T)$ of finite height T, and $f(x,t)$, $a(x,t)$, $b(x,t)$, $c(x,t)$, $\alpha_i(t)$, and $\beta_i(t)$ $(i = 1,2)$ are given functions of $x \in \overline{\Omega}$ and $t \in [0,T]$.

Boundary Value Problem I. *Find a solution $u(x,t)$ to the equation*

$$\begin{gathered} Lu(x,t) \equiv u_{tt}(x,t) - u_{xxtt}(x,t) + a(x,t)u_{xx}(x,t) \\ +b(x,t)u_x(x,t) + c(x,t)u(x,t) = f(x,t) \end{gathered} \tag{1}$$

in Q satisfying

$$\begin{cases} u_x(0,t) = \alpha_1(t)u(0,t) + \alpha_2(t)u(1,t), & 0 < t < 1, \\ u_x(1,t) = \beta_1(t)u(0,t) + \beta_2(t)u(1,t), & 0 < t < 1, \end{cases} \tag{2}$$

$$u(x,0) = u_t(x,0) = 0 \quad \text{for } x \in \Omega. \tag{3}$$

Boundary Value Problem II. *Find a solution $u(x,t)$ to* (1) *in Q satisfying* (3) *as well as the condition*

$$\begin{cases} u(0,t) = \alpha_1(t)u_x(0,t) + \alpha_2(t)u(1,t), & 0 < t < 1, \\ u_x(1,t) = \beta_1(t)u_x(0,t) + \beta_2(t)u(1,t), & 0 < t < 1. \end{cases} \tag{4}$$

Boundary Value Problem III. *Find a solution $u(x,t)$ to* (1) *in Q satisfying* (3) *as well as the condition*

$$\begin{cases} u(0,t) = \alpha_1(t)u_x(0,t) + \alpha_2(t)u_x(1,t), & 0 < t < 1, \\ u(1,t) = \beta_1(t)u_x(0,t) + \beta_2(t)u_x(1,t), & 0 < t < 1. \end{cases} \tag{5}$$

Equation (1) is an analog of the Boussinesq equation arising in the theory of long waves and describing the waves in plasma and the longitudinal waves in a rod

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(see, for example, [11]). Solvability of the boundary value problems and initial-boundary value problems for the equations of the form (1) was studied sufficiently well (see [1–5]) in contrast to the nonlocal problems. We can only note the article [6] devoted to the study of problems with integral conditions of a very particular form and the article [7], where the problems with integral conditions over spatial variables are treated (note that [6] and [7] contain only the existence theorems of generalized solutions).

1. Solvability of Boundary Value Problem I

Let V be the following function space:

$$V = \{v(x,t) : v(x,t) \in L_2\big(0,T;W_2^2(\Omega)\big),\ v_t(x,t) \in L_2\big(0,T;W_2^2(\Omega)\big),\\ v_{tt}(x,t) \in L_2\big(0,T;W_2^2(\Omega)\big)\}.$$

This space is endowed with the natural norm

$$\|v\|_V = \|v\|_{L_2(0,T;W_2^2(\Omega))} + \|v_t\|_{L_2(0,T;W_2^2(\Omega))} + \|v_{tt}\|_{L_2(0,T;W_2^2(\Omega))}.$$

Introduce the notations:

$$F(x,t) = f(x,t) - a(x,t)u_{xx}(x,t) - b(x,t)u_x(x,t) - c(x,t)u(x,t).$$

In the proof of solvability of boundary value problems we use the inequality

$$v^2(y,\tau) \le \delta_0^2 \int\limits_0^1 v_x^2(x,\tau)\,dx + \left(1 + \frac{1}{\delta_0^2}\right) \int\limits_0^1 v^2(x,\tau)\,dx \tag{6}$$

valid for every positive number δ_0 and every point y of $[0,1]$.

Theorem 1. *Assume that*

$$a(x,t) \in L_\infty(Q), \quad b(x,t) \in L_\infty(Q), \quad c(x,t) \in L_\infty(Q), \tag{7}$$

$$\alpha_i(t) \in C^2([0,T]), \quad \beta_i(t) \in C^2([0,T]), \quad i = 1,2, \tag{8}$$

$$\alpha_1(t)\xi^2 + [\alpha_2(t) - \beta_1(t)]\xi\eta - \beta_2(t)\eta^2 \ge 0, \quad t \in [0,T],\ (\xi,\eta) \in \mathbb{R}^2, \tag{9}$$

$$\lambda_0[\alpha_2(t) + \beta_2(t)] \ne 2. \tag{10}$$

Then, for every function $f(x,t) \in L_2(Q)$, Boundary Value Problem I *is uniquely solvable in V.*

Proof. Demonstrate that a solution to Boundary Value Problem I from V satisfies «good» a priori estimates (below these estimates allow us to justify the method of continuation in a parameter). Consider the equality

$$\int\limits_0^t \int\limits_0^1 [u_{\tau\tau}(x,\tau) - u_{xx\tau\tau}(x,\tau) + a(x,\tau)u_{xx}(x,\tau) + b(x,\tau)u_x(x,\tau)\\ + c(x,\tau)u(x,\tau)]u_{\tau\tau}(x,\tau)\,dxd\tau = \int\limits_0^t \int\limits_0^1 f(x,\tau)u_{\tau\tau}(x,\tau)\,dxd\tau.$$

Integrating by parts and taking the boundary conditions and the last equality into account, we come to the next equality

$$\int_0^t\int_0^1 u_{\tau\tau}^2(x,\tau)\,dxd\tau+\int_0^t\int_0^1 u_{x\tau\tau}^2(x,\tau)\,dxd\tau$$

$$+\int_0^t\left[\alpha_1(\tau)u_{\tau\tau}^2(0,\tau)+(\alpha_2(\tau)-\beta_1(\tau))u_{\tau\tau}(1,\tau)u_{\tau\tau}(0,\tau)-\beta_2(\tau)u_{\tau\tau}^2(1,\tau)\right]d\tau$$

$$=\int_0^t[-\alpha_{1\tau\tau}(\tau)u(0,\tau)u_{\tau\tau}(0,\tau)-2\alpha_{1\tau}(\tau)u_\tau(0,\tau)u_{\tau\tau}(0,\tau)-\alpha_{2\tau\tau}(\tau)u(1,\tau)u_{\tau\tau}(0,\tau)$$

$$-2\alpha_{2\tau}(\tau)u_\tau(1,\tau)u_{\tau\tau}(0,\tau)+\beta_{1\tau\tau}(\tau)u(0,\tau)u_{\tau\tau}(1,\tau)+2\beta_{1\tau}(\tau)u_\tau(0,\tau)u_{\tau\tau}(1,\tau)$$

$$+\beta_{2\tau\tau}(\tau)u(1,\tau)u_{\tau\tau}(1,\tau)+2\beta_{2\tau}(\tau)u_\tau(1,\tau)u_{\tau\tau}(1,\tau)]\,d\tau$$

$$+\int_0^t\int_0^1 F(x,\tau)u_{\tau\tau}(x,\tau)\,dxd\tau, \tag{11}$$

where

$$F(x,\tau)=f(x,\tau)-a(x,\tau)u_{xx}(x,\tau)-b(x,\tau)u_x(x,\tau)-c(x,\tau)u(x,\tau).$$

Estimate the right-hand side of (11). The first summand is estimated by using the Young inequality and (6), together with (7) and (8), as follows:

$$\left|\int_0^t \alpha_{1\tau\tau}(\tau)u(0,\tau)u_{\tau\tau}(0,\tau)\,d\tau\right|\le\delta^2\int_0^t\int_0^1\left[u_{x\tau\tau}^2(x,\tau)+u_{\tau\tau}^2(x,\tau)\right]dxd\tau$$

$$+\frac{A}{\delta^2}\int_0^t\int_0^1\left[u_x^2(x,\tau)+u^2(x,\tau)\right]dxd\tau.$$

Here δ is an arbitrary positive number and the number A is determined by the function α_1. Involving the inequality

$$\int_0^t\int_0^1 u^2\,dxd\tau\le T^2\int_0^t\int_0^\tau\int_0^1 u_{\xi\xi}^2\,dxd\xi d\tau,$$

we have

$$\left|\int_0^t \alpha_{1\tau\tau}(\tau)u(0,\tau)u_{\tau\tau}(0,\tau)\,d\tau\right|\le\delta^2\int_0^t\int_0^1\left[u_{x\tau\tau}^2(x,\tau)+u_{\tau\tau}^2(x,\tau)\right]dxd\tau$$

$$+\frac{AT^2}{\delta^2}\int_0^t\int_0^1\left[u_{x\xi\xi}^2(x,\xi)+u_{\xi\xi}^2(x,\xi)\right]dxd\tau.$$

The remaining summands are estimated similarly. Integrating the last summand by parts, we derive the inequality

$$\int_0^t \int_0^1 u_{\tau\tau}^2(x,\xi)\,dxd\tau + \int_0^t \int_0^1 u_{x\tau\tau}^2(x,\xi)\,dxd\tau$$
$$\leq M_1 \int_0^t \int_0^\tau \int_0^1 \left[u_{\xi\xi}^2(x,\xi) + u_{x\xi\xi}^2(x,\xi)\right] dxd\xi d\tau + M_2,$$

where M_1 and M_2 are determined by the functions $\alpha_i(t)$, $\beta_i(t)$, $i = 0,1$, $a(x,t)$, $b(x,t)$, and $c(x,t)$.

The Gronwall lemma yields the estimate

$$\int_0^t \int_0^1 u_{\tau\tau}^2\,dxd\tau + \int_0^t \int_0^1 u_{x\tau\tau}^2\,dxd\tau \leq M_3.$$

To obtain the second estimate, we multiply (1) that is written in the variables x and τ by $-u_{xx\tau\tau}$ and integrate the result with respect to the variable x from 0 to 1 and the variable τ from 0 to t and so

$$\int_0^t \int_0^1 u_{xx\tau\tau}^2\,dxd\tau = \int_0^t \int_0^1 u_{\tau\tau}u_{xx\tau\tau}\,dxd\tau - \int_0^t \int_0^1 Fu_{xx\tau\tau}\,dxd\tau,$$

where

$$F(x,\tau) = f(x,\tau) - a(x,\tau)u_{xx}(x,\tau) - b(x,\tau)u_x(x,\tau) - c(x,\tau)u(x,\tau).$$

The Young and Hölder inequalities and the first estimate imply that

$$\int_0^t \int_0^1 u_{xx\tau\tau}^2\,dxd\tau \leq M_4 \int_0^t \int_0^\tau \int_0^1 u_{xx\xi\xi}^2\,dxd\xi d\tau + M_5.$$

Here the constants M_4 and M_5 are defined by the functions $\alpha_i(t)$, $\beta_i(t)$, $i = 0,1$, $a(x,t)$, $b(x,t)$, $c(x,t)$, and the number T.

This inequality and the Gronwall lemma ensure the second a priori estimate

$$\int_0^t \int_0^1 u_{xx\tau\tau}^2\,dxd\tau \leq M_5 e^{M_4} \leq M_6.$$

The above estimates validate the conclusion that the norm of $u(x,t)$ in V is bounded; i.e.,

$$\|u\|_V \leq M_7, \tag{12}$$

where the constant M_7 is determined by $\alpha_i(t), \beta_i(t)$, $i = 0,1$, $a(x,t), b(x,t)$, $c(x,t)$, and the number T.

To prove solvability of the boundary value problem, we employ the method of continuation in a parameter. Given a number λ in $[0,1]$, examine the following family of boundary value problems: *Find a solution $u(x,t)$ to* (1) *in Q satisfying* (3) *as well as the condition*

$$\begin{cases} u_x(0,t) = \lambda[\alpha_1(t)u(0,t) + \alpha_2(t)u(1,t)], & 0 < t < 1, \\ u_x(1,t) = \lambda[\beta_1(t)u(0,t) + \beta_2(t)u(1,t)], & 0 < t < 1. \end{cases} \tag{2_λ}$$

Denote by Λ the set of numbers λ from $[0,1]$ for which problem (1), (2_λ), (3) is solvable in V for every $f(x,t)$ in $L_2(Q)$. As is known, if λ is nonempty, open, and closed then it coincides with $[0,1]$ (see [8]). Coincidence of Λ with $[0,1]$ means that the boundary value problem (1)–(3) is solvable in V.

The set Λ is obviously nonempty, since 0 belongs to this set. To prove that Λ is open, it suffices to demonstrate that if $\lambda_0 \in \Lambda$ then the numbers $\lambda = \lambda_0 + \tilde{\lambda}$ also belong Λ provided that the quantity $|\tilde{\lambda}|$ is sufficiently small. Let $v(x,t)$ be a given function in V. Put

$$\varphi(t) = \tilde{\lambda}[\alpha_1(t)v(0,t) + \alpha_2(t)v(1,t)],$$
$$\psi(t) = \tilde{\lambda}[\beta_1(t)v(0,t) + \beta_2(t)v(1,t)],$$
$$u_0(x,t) = \frac{[(1-\lambda_0\beta_2(t))x^2 - (2-\lambda_0\beta_2(t))x]\varphi(t) - [(1-\lambda_0\alpha_2(t))x^2 + \lambda_0\alpha_2(t)x]\psi(t)}{\lambda_0[\alpha_2(t)+\beta_2(t)] - 2}$$

and

$$g_v(x,t) = f(x,t) - Lu_0(x,t).$$

Consider the following auxiliary problem: *Find a solution $w(x,t)$ to the equation*

$$Lw(x,t) = g_v(x,t) \tag{1$'$}$$

in Q satisfying (2_{λ_0}) *and* (3).

Since $v(x,t)$ lies in V, we have $Lu_0 \in L_2(Q)$ and a solution to this problem also belongs to this space. Hence, we can define the operator Φ taking V into itself, $\Phi(w) = v$.

Assume that $v_1(x,t)$ and $v_2(x,t)$ are functions in V, while $w_1(x,t)$ and $w_2(x,t)$ are solutions to $(1')$, (2_{λ_0}), (3) with the respective functions $g_{v_1}(x,t)$ and $g_{v_2}(x,t)$. Repeating the proof of (12), we arrive at the inequality

$$\|\Phi(w_1) - \Phi(w_2)\|_V \leq |\tilde{\lambda}| M_8 \|v_1 - v_2\|_{L_2(Q)}.$$

The operator Φ is contractive if the number $\tilde{\lambda}$ is so small that $|\tilde{\lambda}|M_8 \leq 1$. A fixed point of this operator is a solution $w(x,t)$ to the boundary value problem.

Given $g_w(x,t)$, define $u(x,t)$ as follows:

$$u(x,t) = w(x,t) + \frac{\tilde{\lambda}[(1-\lambda_0\beta_2(t))x^2 - (2-\lambda_0\beta_2(t))x][\alpha_1(t)w(0,t) + \alpha_2(t)w(1,t)]}{\lambda_0[\alpha_2(t)+\beta_2(t)] - 2}$$
$$+ \frac{\tilde{\lambda}[(1-\lambda_0\alpha_2(t))x^2 + \lambda_0\alpha_2(t)x][\beta_1(t)w(0,t) + \beta_2(t)w(1,t)]}{\lambda_0[\alpha_2(t)+\beta_2(t)] - 2}.$$

The function $u(x,t)$ in V is a solution to (1), (2_λ), (3) for $\lambda = \lambda_0 + \tilde{\lambda}$. The latter means that the number $\lambda_0 + \tilde{\lambda}$ belongs to Λ and Λ is open.

Demonstrate that (12) implies the closedness of Λ. Assume that λ_m is a sequence of numbers from Λ such that $\lambda_m \to \lambda_0$ as $m \to \infty$ and $u_m(x,t)$ is a sequence of solutions to (1), (2_{λ_m}), (3). Put $w_{mk}(x,t) = u_m(x,t) - u_k(x,t)$. The function $w_{mk}(x,t)$ meets the equalities

$$Lw_{mk} = 0,$$
$$\begin{cases} w_{mkx}(0,t) &= \lambda_k[\alpha_1(t)w_{mk}(0,t) + \alpha_2(t)w_{mk}(1,t)] \\ &\quad +(\lambda_m - \lambda_k)[\alpha_1(t)u_m(0,t) + \alpha_2(t)u_m(1,t)], \\ w_{mkx}(1,t) &= \lambda_k[\beta_1(t)w_{mk}(0,t) + \beta_2(t)w_{mk}(1,t)] \\ &\quad +(\lambda_m - \lambda_k)[\beta_1(t)u_m(0,t) + \beta_2(t)u_m(1,t)], \end{cases}$$
$$w_{mk}(x,0) = w_{mkt}(x,0) = 0 \quad \text{for } x \in \Omega.$$

Consequently, the sequence $u_{mk}(x,t)$ is fundamental in V. Therefore, there exists $u(x,t) \in V$ such that $u_{mk}(x,t) \to u(x,t)$ as $m \to \infty$ in V. The function $u(x,t)$ is a solution to (1) satisfying (2_{λ_0}) and (3). Thus, λ_0 belongs to Λ. So the limit points of Λ belong to Λ, and so Λ is closed.

Hence, Λ is nonempty, open, and closed. Therefore, Λ coincides with $[0,1]$ and the boundary value problem (1)–(3) is solvable in V.

2. Solvability of Boundary Value Problem II

Theorem 2. *Assume that the conditions* (7)–(9) *of Theorem* 1 *hold and*

$$\lambda_0^2[\alpha_1(t)\beta_2(t) - \alpha_2(t)\beta_1(t)] + \lambda_0[\alpha_2(t) + \beta_1(t) + \beta_2(t)] \neq 1.$$

Then, for every $f(x,t) \in L_2(Q)$, Boundary Value Problem II *is uniquely solvable in V.*

Proof. First of all, we observe that for the solutions to Problem II satisfies (12) (we verbatim repeat the arguments of the proof of Theorem 1). Next, we use the method of continuation in a parameter again. Let λ belong to $[0,1]$.

Examine the following family of boundary value problems: *Find a solution $u(x,t)$ to* (1) *in Q satisfying* (3) *and such that*

$$\begin{cases} u(0,t) = \lambda[\alpha_1(t)u_x(0,t) + \alpha_2(t)u(1,t)], & 0 < t < 1, \\ u_x(1,t) = \lambda[\beta_1(t)u_x(0,t) + \beta_2(t)u(1,t)], & 0 < t < 1. \end{cases} \tag{4_λ}$$

Denote by Λ the set of numbers λ from $[0,1]$ such that (1), (4_λ), (3) is solvable in V for every function $f(x,t)$ in $L_2(Q)$. As it was said before, if Λ is nonempty, open, and closed then it coincides with $[0,1]$. This coincidence of Λ with $[0,1]$ means that the boundary value problem (1), (3), (4) is solvable in V.

Obviously, Λ is nonempty, since λ contains 0. To prove that Λ is open, it suffices to establish that $\lambda = \lambda_0 + \tilde{\lambda}$ belong to Λ together with $\lambda_0 \in \Lambda$ for sufficiently small $|\tilde{\lambda}|$.

Let $v(x,t)$ be an arbitrary function in V. Put

$$\varphi_1(t) = \tilde{\lambda}[\alpha_1(t)v_x(0,t) + \alpha_2(t)v(1,t)],$$

$$\psi_1(t) = \tilde{\lambda}[\beta_1(t)v_x(0,t) + \beta_2(t)v(1,t)],$$

$$\bar{u}_0(x,t) = \frac{[\lambda_0(\beta_1(t) + (1-x)\beta_2(t)) - 1]\varphi_1(t) - [x + \lambda_0[\alpha_1(t) + (1-x)\alpha_2(t)]\psi_1(t)}{\lambda_0^2[\alpha_1(t)\beta_2(t) - \alpha_2(t)\beta_1(t)] + \lambda_0[\alpha_2(t) + \beta_1(t) + \beta_2(t)]] - 1}.$$

Put $\bar{g}_v(x,t) = f(x,t) - L\bar{u}_0(x,t)$. Consider the following auxiliary problem: *Find a solution $\overline{w}(x,t)$ to the equation*

$$L\overline{w}(x,t) = \bar{g}_v(x,t) \tag{$1''$}$$

in Q satisfying (4_{λ_0}) *and* (3).

Since $v(x,t)$ belongs to V; therefore, $L\bar{u}_0 \in L_2(Q)$ and a solution to this problem also lies in this space. Hence, we can define the operator Φ taking V into itself, $\Phi(\overline{w}) = v$.

Assume that $v_1(x,t)$ and $v_2(x,t)$ are two functions in V and $\overline{w}_1(x,t)$ and $\overline{w}_2(x,t)$ are solutions to ($1''$), (4_{λ_0}), (3) with the functions $\bar{g}_{v_1}(x,t)$ and $\bar{g}_{v_2}(x,t)$, respectively. Repeating the proof of (12), we arrive at the inequality

$$\|\Phi(\overline{w}_1) - \Phi(\overline{w}_2)\|_V \leq |\widetilde{\lambda}| M_9 \|v_1 - v_2\|_{L_2(Q)}.$$

If $\tilde{\lambda}$ is so small that $|\tilde{\lambda}|M_9 \leq 1$ then Φ is contractive. A fixed point $\overline{w}(x,t)$ of this operator is a solution to the boundary value problem.

Given a function $\bar{g}_{\overline{w}(x,t)}$, define the function $u(x,t)$ as follows:

$$u(x,t) = \overline{w}(x,t) + \frac{\tilde{\lambda}[\lambda_0(\beta_1(t) + (1-x)\beta_2(t)) - 1][\alpha_1(t)w_x(0,t) + \alpha_2(t)w(1,t)]}{\lambda_0^2[\alpha_1(t)\beta_2(t) - \alpha_2(t)\beta_1(t)] + \lambda_0[\alpha_2(t) + \beta_1(t) + \beta_2(t)]] - 1}$$
$$+ \frac{\tilde{\lambda}[x + \lambda_0[\alpha_1(t) + (1-x)\alpha_2(t)][\beta_1(t)w_x(0,t) + \beta_2(t)w(1,t)]}{\lambda_0^2[\alpha_1(t)\beta_2(t) - \alpha_2(t)\beta_1(t)] + \lambda_0[\alpha_2(t) + \beta_1(t) + \beta_2(t)]] - 1}.$$

The function $u(x,t)$ belongs to V and serves as a solution to (1), (4_λ), (3) for $\lambda = \lambda_0 + \tilde{\lambda}$. The latter means that $\lambda_0 + \tilde{\lambda}$ belongs to Λ and Λ is open.

We now prove that (12) implies the closedness of Λ. Assume that λ_m is a sequence of numbers in Λ such that $\lambda_m \to \lambda_0$ as $m \to \infty$ and $u_m(x,t)$ is a sequence of solutions to (1), (4_{λ_m}), (3). Put $\overline{w}_{mk}(x,t) = u_m(x,t) - u_k(x,t)$. The functions $\overline{w}_{mk}(x,t)$ meet the equalities

$$L\overline{w}_{mk} = 0,$$

$$\begin{cases} \overline{w}_{mk}(0,t) & = \lambda_k[\alpha_1(t)\overline{w}_{mkx}(0,t) + \alpha_2(t)\overline{w}_{mk}(1,t)] \\ & \quad +(\lambda_{mx} - \lambda_k)[\alpha_1(t)u_m(0,t) + \alpha_2(t)u_m(1,t)], \\ \overline{w}_{mkx}(1,t) & = \lambda_k[\beta_1(t)\overline{w}_{mkx}(0,t) + \beta_2(t)\overline{w}_{mk}(1,t)] \\ & \quad +(\lambda_m - \lambda_k)[\beta_1(t)u_{mx}(0,t) + \beta_2(t)u_m(1,t)], \end{cases}$$

$$\overline{w}_{mk}(x,0) = \overline{w}_{mkt}(x,0) = 0 \quad \text{for } x \in \Omega.$$

Hence, the sequence $u_{mk}(x,t)$ is fundamental in V and so there exists a function $u(x,t) \in V$ such that $u_{mk}(x,t) \to u(x,t)$ as $m \to \infty$ in V. The function $u(x,t)$ satisfies (1) as well as (4_{λ_0}) and (3). Thus, λ_0 belongs to Λ. Since the limit points of Λ belong to Λ; therefore, Λ is closed.

Thus, Λ is nonempty, open, and closed. Hence, Λ coincides with $[0,1]$ and the boundary value problem (1), (3), (4) is solvable in V.

3. Solvability of Boundary Value Problem III

Theorem 3. *Assume that the conditions* (7)–(9) *of Theorem* 1 *hold and*

$$\lambda_0[\beta_1(t) + \beta_2(t) - \alpha_1(t) - \alpha_2(t)] \neq 1.$$

Then, for every $f(x,t) \in L_2(Q)$, Boundary Value Problem III *is uniquely solvable in V.*

Proof. Argue similarly to the proof of Theorems 1 and 2 with the only distinction in the choice of the functions

$$\varphi_2(t) = \tilde{\lambda}[\alpha_1(t)v_x(0,t) + \alpha_2(t)v_x(1,t)],$$

$$\psi_2(t) = \tilde{\lambda}[\beta_1(t)v_x(0,t) + \beta_2(t)v_x(1,t)],$$

$$\widetilde{u}_0(x,t) = \frac{[x - 1 + \lambda_0(\beta_1(t) + \beta_2(t))]\varphi_2(t) - [1 + \lambda_0[\alpha_1(t) + \alpha_2(t)]\psi_2(t)}{\lambda_0[\beta_1(t) + \beta_2(t) - \alpha_1(t) - \alpha_2(t)] - 1},$$

which are used to prove the openness of the corresponding set Λ.

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A BOUNDARY VALUE PROBLEM FOR A MIXED TYPE EQUATION WITH A SPECTRAL PARAMETER

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Abstract. In a cylindrical domain of the space $\mathbb{R}^{n+1}$ we study Vragov's boundary value problem for a mixed type equation of the second order with a spectral parameter. Under certain conditions on the coefficients, we establish the a priori estimates that allow us to prove a unique solvability of this boundary value problem in the energy space. Some sufficient conditions are obtained for the Fredholm solvability of the boundary value problem in this space.

Keywords: mixed type equation, a priori estimate, inequality, equality, orthogonality conditions

At present, there are many articles studying boundary value problems for mixed type equations with a spectral parameter [1–7]. The most complete bibliography on this topic can be found in [2, 5, 6]. But the most part of the results is obtained for the modal equations of mixed type on a plane.

In this article we study a boundary value problem for a mixed type equation with a spectral parameter in the multidimensional case. For the first time, this problem was studied by V. N. Vragov in [3, 4, 8].

Assume that Ω is a bounded domain in $\mathbb{R}^n$ with boundary $S \in C^1$, $\Omega_t = \Omega \times t$ for $0 \le t \le T$, $S_T = S \times (0,T)$, and $Q = \Omega \times (0,T)$.

In the cylindrical domain Q we consider the mixed type equation

$$Lu - \lambda u = f(x,t), \tag{1}$$

where

$$Lu \equiv k(x,t)u_{tt} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{ij}(x)u_{x_i}) + a(x,t)u_t + c(x)u$$

and λ is a complex number.

We suppose that the coefficients of the differential operator L are sufficiently smooth functions in $\overline{Q}$ and satisfy the conditions

$$a_{ij} = a_{ji}, \quad \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \ge \nu|\xi|^2, \quad \xi \in \mathbb{R}^n,\ \nu > 0.$$

Introduce the sets

$$P_0^{\pm} = \{(x,0) : k(x,0) \gtrless 0,\ x \in \Omega\}, \quad P_T^{\pm} = \{(x,T) : k(x,T) \gtrless 0,\ x \in \Omega\}.$$

Boundary Value Problem. *Find a solution to* (1) *in* Q *such that*

$$u|_{S_T} = 0, \quad u|_{t=0} = 0, \quad u_t|_{\overline{P}_0^+} = 0, \quad u_t|_{\overline{P}_T^-} = 0. \tag{2}$$

Let C_L be the class of complex functions from $W_2^2(Q)$ which satisfy (2). Denote by C_{L^*} the class of complex functions from $W_2^2(Q)$ satisfying the adjoint boundary conditions

$$v|_{S_T} = 0, \quad v|_{\overline{P}_0^-} = 0, \quad v|_{\overline{P}_T^+} = 0, \quad [kv_t + (k_t - a)v]|_{t=T} = 0, \tag{2*}$$

and

$$L^*v \equiv k(x,t)v_{tt} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)v_{x_j}) + (2k_t - a)v_t + (c + k_{tt} - a_t)v.$$

Endow the Sobolev space $W_2^1(Q)$ with the inner product

$$(u,v)_1 = \int\limits_Q \Big[u\bar{v} + \sum_{i=1}^{n} u_{x_i}\bar{v}_{x_i} + u_t\bar{v}_t\Big]\,dQ, \quad \|u\|_1^2 = (u,u)_1, \quad u,v \in W_2^1(Q);$$

for the $L_2(Q)$ space we have

$$(u,v) = \int\limits_Q u\bar{v}\,dQ, \quad \|u\|^2 = (u,u), \quad u,v \in L_2(Q).$$

Assume that $\widetilde{W}_2^1(Q)$ is the completion of C_L in the norm of the Sobolev space $W_2^1(Q)$, and $\widehat{W}_2^1(Q)$ is a subspace of $W_2^1(Q)$ comprising the functions that satisfy the conditions

$$v|_{S_T} = 0, \quad v|_{\overline{P}_0^-} = 0, \quad v|_{\overline{P}_T^+} = 0.$$

Definition 1. A function $u \in L_2(Q)$ is called a *strong solution to* (1), (2) if there exists a sequence of functions $u_m \in C_L$ such that

$$\lim_{m\to\infty} \|u_m - u\| = \lim_{m\to\infty} \|Lu_m - \lambda u_m - f\| = 0, \quad f \in L_2(Q).$$

Definition 2. A function $u(x,t) \in \widetilde{W}_2^1(Q)$ is called *a generalized solution to* (1), (2) if

$$a(u,v) - \lambda(u,v) = (f,v) \tag{3}$$

for every function $v \in \widehat{W}_2^1(Q)$, $f \in L_2(Q)$ and

$$a(u,v) \equiv \int\limits_Q \Big[-ku_t\bar{v}_t + \sum_{i,j=1}^{n} a_{ij}u_{x_i}\bar{v}_{x_j} + (a - k_t)u_t\bar{v} + cu\bar{v}\Big]\,dQ.$$

Similarly, we can define strong and generalized solutions to the adjoint boundary value problem for the equation $L^*v - \bar{\lambda}v = g$, $g \in L_2(Q)$ with the boundary conditions (2*).

In view of the equality

$$(Lu,v) = (u,L^*v), \quad u \in C_L,\ v \in C_{L^*},$$

the operator L with the domain C_L admits the closure $\overline{L} = A$ in $L_2(Q)$. Introduce the set

$$D(\delta,\gamma,\mu) = \Big\{\lambda \in C : \operatorname{Re}\lambda \le \mu - \frac{1}{\delta\gamma}(\operatorname{Im}\lambda)^2\Big\}.$$

Lemma 1. *Assume that*

$$c(x) \geq c_0, \quad a - \frac{1}{2}k_t + \gamma k \geq \delta > 0, \quad \gamma > 0.$$

Then

$$\|u\|_1 \leq c_1 \|Au - \lambda u\|, \quad c_1 > 0, \tag{4}$$

for $\lambda \in D(\delta, \gamma, c_0)$ and all $u \in D(A - \lambda E)$.

Proof. It suffices to establish (4) for functions $u(x,t)$ from C_L. Given $u \in C_L$, integrating by parts and taking account of (2), we infer

$$\operatorname{Re} \int\limits_Q (Lu - \lambda u) e^{-2\gamma t} \bar{u}_t \, dQ = \int\limits_Q e^{-2\gamma t} \bigg[\Big(a - \frac{1}{2}k_t + \gamma k \Big) |u_t|^2$$

$$+ \gamma \sum_{i,j=1}^n a_{ij} u_{x_i} \bar{u}_{x_j} + (-\operatorname{Re}\lambda + c)\gamma |u|^2 - (\operatorname{Im}\lambda) \operatorname{Im}(u \bar{u}_t) \bigg] dQ + \frac{1}{2} \int\limits_{P_T^+} k e^{-2\gamma T} |u_t|^2 \, dx$$

$$- \frac{1}{2} \int\limits_{P_0^-} k |u_t|^2 \, dx + \frac{e^{-2\gamma T}}{2} \int\limits_{\Omega_T} \bigg[\sum_{i,j=1}^n a_{ij} u_{x_i} \bar{u}_{x_j} + (-\operatorname{Re}\lambda + c)|u|^2 \bigg] dx.$$

Next, the Cauchy inequality with ε and the Friedrichs–Poincaré inequality yield (4). Lemma 1 is proven.

Inequality (4) implies that $D(A - \lambda E) \subseteq \widetilde{W}_2^1(Q)$. Consider $D^*(\delta_2, \gamma, \mu_1, \mu_2) = \big\{ \lambda \in C : \operatorname{Re}\lambda \leq -\frac{1}{\gamma}\mu_1 - \frac{1}{\delta_2 \gamma} e^{4\gamma T} (\operatorname{Im}\lambda)^2,\ \operatorname{Re}\lambda \leq -\mu_2 \big\}$, $\delta_2 > 0$, $\mu_1 > 0$, $\mu_2 \geq 0$.

Lemma 2. *Assume that*

$$a - \frac{3}{2}k_t + \gamma k \geq \delta > 0, \quad \gamma > 0,$$

and either

$$k(x,0) \geq 0,\ k(x,T) < 0, \quad \text{or } k(x,0) < 0,\ k(x,T) > 0,$$

$$\text{or } k(x,0) < 0,\ k(x,T) < 0, \quad \text{or } k(x,0) \geq 0,\ k(x,T) > 0.$$

Then, for $\lambda \in D^*(\delta_2, \gamma, \mu_1, \mu_2)$, we have the inequality

$$\|v\|_1 \leq c_2 \|A^* v - \bar{\lambda} v\|, \quad c_2 > 0, \tag{5}$$

for all $v \in D(A^* - \bar{\lambda} E)$, where the parameters δ_2, μ_1, and μ_2 depend on the case chosen.

Proof. Given $v(x,t)$ in C_{L^*}, examine the expression

$$I \equiv \operatorname{Re} \int\limits_Q e^{2\gamma t} (L^* v - \bar{\lambda} v)(-\xi \bar{v}_t + \eta \bar{v}) \, dQ,$$

where $\xi(t), \eta(t)$ are nonnegative infinitely differentiable functions which are chosen appropriately below. Integrating by parts, we have

$$I = \int\limits_Q e^{2\gamma t}\bigg\{\bigg[\Big(a - \frac{3}{2}k_t + \gamma k\Big)\xi - k\Big(\eta - \frac{1}{2}\xi_t\Big)\bigg]|v_t|^2$$

$$+\Big(\eta + \frac{1}{2}\xi_t + \gamma\xi\Big)\sum_{i,j=1}^n a_{ij}v_{x_i}\bar v_{x_j} + \bigg[\Big(-\operatorname{Re}\lambda\Big(\eta + \frac{1}{2}\xi_t + \gamma\xi\Big) + (c + k_{tt} - a_t)\eta\Big)\bigg]|v|^2$$

$$+\xi\operatorname{Im}\lambda\operatorname{Im}(v_t\bar v) + [(k_t - a - \gamma k)\eta - k\eta_t]\operatorname{Re}(v_t\bar v) - (c + k_{tt} - a_t)\xi\operatorname{Re}(v_t\bar v)\bigg\}\,dQ$$

$$-\frac{1}{2}\int\limits_\Omega ke^{2\gamma t}\xi|v_t|^2\,dx\bigg|_{t=0}^{t=T} - \frac{1}{2}\int\limits_\Omega e^{2\gamma t}\xi\bigg[\sum_{i,j=1}^n a_{ij}\bar v_{x_i}v_{x_j} - \operatorname{Re}\lambda|v|^2\bigg]\,dx\bigg|_{t=0}^{t=T}$$

$$+\int\limits_\Omega ke^{2\gamma t}\eta\operatorname{Re}(v_t\bar v)\,dx\bigg|_{t=0}^{t=T}. \tag{6}$$

1. Let $k(x,0) \ge 0$ and $k(x,T) < 0$. Choose a number T_0 so that $k(x,T) \le -\delta_1 < 0$, $t \in [T_0, T]$, $T_0 < T$.

We assume that $\xi(t) = 1$, $t \in [0, T_0]$, and $\xi_t \le 0$, $\xi(T) = 0$. Put $\eta(t) = -\frac{1}{2}\xi_t + \gamma$. In this case Lemma 2 yields

$$\Big(a - \frac{3}{2}k_t + \gamma k\Big)\xi - k\Big(\eta - \frac{1}{2}\xi_t\Big) \ge \delta_2 = \min\{\delta, \delta_1\gamma, \nu\gamma\}.$$

The boundary conditions (2*) and the embedding theorems validate the inequality

$$\bigg|\int\limits_{\Omega_T} ke^{2\gamma T}\eta\operatorname{Re}(v_t\bar v)\,dx\bigg| \le \varepsilon_1\int\limits_Q\bigg[|v_t|^2 + \sum_{i=1}^n|v_{x_i}|^2\bigg]\,dQ + C_{\varepsilon_1}\int\limits_Q|v|^2\,dQ, \tag{7}$$

where $\varepsilon_1 = \frac{1}{2}\delta_2$. On the other hand,

$$\bigg|\int\limits_{\Omega_0} k(x,0)\eta(0)\operatorname{Re}(v_t\bar v)\,dx\bigg| \le \frac{1}{4}\int\limits_{\Omega_0} k(x,0)|v_t|^2\,dx + \int\limits_{\Omega_0} k(x,0)\eta^2(0)|v|^2\,dx. \tag{8}$$

From (7), (8), and (6) we see that

$$I \ge \int\limits_Q\bigg\{\frac{\delta_2}{8}|v_t|^2 + \frac{1}{2}\nu\gamma\sum_{j=1}^n|v_{x_j}|^2 + \bigg[(-\operatorname{Re}\lambda)\gamma - \frac{1}{\delta_2}e^{4\gamma t}(\operatorname{Im}\lambda)^2 - \mu_1\bigg]|v|^2\bigg\}\,dQ$$

$$+\frac{1}{2}\int\limits_{\Omega_0}(-\operatorname{Re}\lambda - \mu_2)|v|^2\,dx, \tag{9}$$

where

$$\mu_1 = \max_{\overline Q}\{(2/\delta_2)[|k_t - a - \gamma k|\eta$$

$$+|k\eta_t| + |c + k_{tt} - a_t|]^2 e^{4\gamma t} + e^{2\gamma t}|c + k_{tt} - a_t|\eta\} + C_{\varepsilon_1},$$

$$\mu_2 = 2\max_{\overline\Omega}(k(x,0)\eta^2(0)).$$

To derive the a priori estimate (5), we employ (9), as well as Friedrichs–Poincaré and Hölder inequalities.

2. Let $k(x,0) < 0$ and $k(x,T) > 0$. Take a number t_0 such that

$$k(x,t) \le -\delta_1 < 0, \quad t \in [0,t_0], \quad 0 < t_0 < T.$$

Let $\xi(t)$ satisfy the conditions

$$\xi(0) = 0, \quad \xi_t \ge 0, \quad \xi(t) = 1, \quad t \in [t_0, T].$$

Put $\eta_t = \frac{1}{2}\xi_t + \gamma$. Involving the Cauchy inequality with ε and (6), we see that

$$I \ge \int\limits_Q \left\{ \frac{\delta_2}{4}|v_t|^2 + \nu\gamma \sum_{j=1}^n |v_{x_j}|^2 + \left[(-\operatorname{Re}\lambda)\gamma - \frac{1}{\delta_2} e^{4\gamma t} (\operatorname{Im}\lambda)^2 - \mu_1 \right] |v|^2 \right\} dQ, \quad (10)$$

where $\delta_2 = \min\{\delta, \delta_1\gamma\}$ and $\mu_2 = 0$. This inequality ensures the a priori estimate of Lemma 2.

3. For $k(x,0) < 0$ and $k(x,T) < 0$, we assume that

$$k(x,t) \le -\delta_1 < 0, \quad t \in [0,t_0] \cup [T_0, T].$$

Let $\xi(t)$ satisfy the relations

$$\xi(0) = 0, \quad \xi_t \ge 0,\ t \in [0,t_0], \quad \xi(t) = 1,\ t \in [t_0, T_0], \quad \xi_t \le 0,\ t \in [T_0, T], \quad \xi(T) = 0.$$

Put $\eta_t = \frac{1}{2}\xi_t + \gamma$ for $0 \le t \le t_0$, $\eta(t) = \gamma$ for $t \in [t_0, T_0]$, and $\eta(t) = -\frac{1}{2}\xi_t + \gamma$ for $T_0 \le t \le T$. The conditions of Lemma 2 imply that

$$\left(a - \frac{3}{2}k_t + \gamma k \right) - k\left(\eta - \frac{1}{2}\xi_t \right) \ge \delta_2 = \min\{\delta, \delta_1\gamma, \nu\gamma\}.$$

Note that (6) and (7) validate (9) without the boundary integral over Ω_0, $\mu_2 = 0$. Inequality (9) ensures (5).

4. Assume that $k(x,0) \ge 0$ and $k(x,T) > 0$. To justify the a priori estimate of Lemma 2, it suffices to consider the functions $\xi(t) \equiv 1$ and $\eta(t) \equiv 0$.

In this case (6) yields (10), where $\delta_2 = \delta$, $\mu_2 = 0$, and

$$\mu_1 = \frac{e^{4\gamma T}}{2\delta} \max_{\overline{Q}} |c + k_{tt} - a_t|^2.$$

Lemma 2 is proven.

Theorem 1. *Assume that the conditions of Lemmas* 1 *and* 2 *are fulfilled and* $\lambda \in D(\delta, \gamma, c_0) \cap D^*(\delta_2, \gamma, \mu_1, \mu_2)$. *Then, for every* $f \in L_2(Q)$, *there exists a unique strong solution in* $D(A - \lambda E)$ *to* (1), (2).

Proof. The a priori estimate (5) implies that $N(A^* - \bar{\lambda}E) = 0$. As a direct consequence, we have $\overline{R(A - \lambda E)} = L_2(Q)$. On the other hand, the a priori estimate (4) yields $R(A - \lambda E) = \overline{R(A - \lambda E)}$. Hence, the equation $Au - \lambda u = f$ is always solvable. The uniqueness results from Lemma 1. Theorem 1 is proven.

Note that a strong solution in $D(A - \lambda E)$ to (1), (2) whose existence is provided by Theorem 1 is a generalized solution in $\widetilde{W}_2^1(Q)$ to (1), (2).

The boundary value problem adjoint to (1), (2) is of the form

$$L^* v - \bar{\lambda} v = g(x,t), \quad (x,t) \in Q, \quad (1^*)$$

with the boundary conditions (2*).

Theorem 2. *Assume that the conditions of Lemmas* 1 *and* 2 *are fulfilled and* $\lambda \in D(\delta, \gamma, c_0) \cap D^*(\delta_2, \gamma, \mu_1, \mu_2)$. *Then, for every* $g \in L_2(Q)$, *there exists a unique strong solution in* $D(A^* - \bar{\lambda}E)$ *to* (1*), (2*).

The proof of Theorem 2 is in line with that of Theorem 1. Note that the results of Theorem 1 in the real case were obtained by V. N. Vragov in [8] under some stronger requirements on the coefficients of (1) and the surface S.

Following [2], we introduce the energy classes

$$V_L^1(Q) = D(A - \lambda E), \quad V_{L^*}^1(Q) = D(A^* - \bar{\lambda}E)$$

for the operators $A_\lambda = A - \lambda E$ and $A_\lambda^* = A^* - \bar{\lambda}E$.

Lemma 3. *Let the conditions of Theorem* 1 *hold. Then* $\left(A_\lambda^{-1}\right)^* = (A_\lambda^*)^{-1}$.

Proof. Theorems 1 and 2 imply that

$$(A_\lambda u, v) = (u, A_\lambda^* v), \quad u \in V_L^1(Q), \ v \in V_{L^*}^1(Q).$$

Put $\varphi = A_\lambda u$ and $\psi = A_\lambda^* v$. In view of Theorems 1 and 2, we have the form

$$(\varphi, (A_\lambda^*)^{-1}\psi) = \left(A_\lambda^{-1}\varphi, \psi\right) = \left(\varphi, \left(A_\lambda^{-1}\right)^*\psi\right), \quad \varphi, \psi \in L_2(Q),$$

and so $(A_\lambda^*)^{-1}\psi = \left(A_\lambda^{-1}\right)^*\psi$, $\psi \in L_2(Q)$. Lemma 3 is proven.

Theorem 3. *Under the conditions of Lemmas* 1 *and* 2, *the following hold:*

1. *The boundary value problem* (1), (2) *is uniquely solvable in* $V_L^1(Q)$ *except at most countably many points* $\{\lambda_k\}$ *with the only possible limit point* $\lambda = \infty$. *Together with* λ_k, *the spectrum of* (1), (2) *contains* $\bar{\lambda}_k$. *For* $\lambda = \lambda_k$ *in the spectrum of* (1), (2), *the homogeneous boundary value problem* (1), (2) *has a nontrivial solutions in* $V_L^1(Q)$ *and to every* λ_k, *there correspond finitely many* n_k *of linearly independent solutions.*

2. *The numbers* $\{\lambda_k\}$ *and* $\{\bar{\lambda}_k\}$ *are the eigenvalues of the adjoint boundary value problem* (1*), (2*); *moreover,* $\bar{\lambda}_k$ *is of multiplicity* n_k *and the corresponding eigenfunction* v_{k_j}, $j = \overline{1, n_k}$, *lies in* $V_{L^*}^1$. *For* (1), (2) *with* $\lambda = \lambda_k$ *to be solvable, it is necessary and sufficient that the orthogonality conditions* $(f, v_{k_j}) = 0$, $j = \overline{1, n_k}$, *be valid.*

Proof. The boundary value problem (1), (2) is equivalent to the operator equation

$$A_{\lambda_0} u \equiv Au - \lambda_0 u = (\lambda - \lambda_0)u + f. \tag{11}$$

In case $\lambda_0 \in D(\delta, \gamma, c_0) \cap D^*(\delta_2, \gamma, \mu_1, \mu_2)$, the operator A_{λ_0} has bounded inverse $A_{\lambda_0}^{-1}$ acting from $L_2(Q)$ into $V_L^1(Q)$ by Theorem 1. Hence, (11) is equivalent to the equation

$$u = (\lambda - \lambda_0)A_{\lambda_0}^{-1}u + A_{\lambda_0}^{-1}f. \tag{12}$$

Since the embedding of $V_L^1(Q)$ in $L_2(Q)$ is compact, we conclude that $A_{\lambda_0}^{-1}$ as an operator from $L_2(Q)$ into $L_2(Q)$ is compact as well. So claim 1 of Theorem 3 is proved.

Note that the adjoint problem (1*), (2*) is equivalent to the operator equation

$$v = (\bar{\lambda} - \bar{\lambda}_0)(A_{\lambda_0}^*)^{-1}v + (A_{\lambda_0}^*)^{-1}g. \tag{12*}$$

By Lemma 3 the homogeneous equation (12*) is adjoint to the homogeneous equation (12).

For (12) for $\lambda = \lambda_k$ to be solvable, it is necessary and sufficient that

$$\left(A_{\lambda_0}^{-1} f, v_{k_j}\right) = 0, \quad j = \overline{1, n_k}.$$

Hence,

$$0 = \left(f, \left(A_{\lambda_0}^{-1}\right)^* v_{k_j}\right) = \left(f, (A_{\lambda_0}^*)^{-1} v_{k_j}\right) = \frac{1}{\bar{\lambda}_k - \bar{\lambda}_0} (f, v_{k_j}).$$

The proof of Theorem 3 is complete.

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A CONJUGATION PROBLEM FOR SOME HIGHER ORDER NONCLASSICAL EQUATIONS. II

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Abstract. This is a continuation of the authors' article [1] which is devoted to solvability of conjugate problems (generalized diffraction problems) for some nonclassical higher order differential equations of composite type. We prove existence and uniqueness theorems of regular solutions to these problems.

Keywords: higher order differential equation of composite type, conjugation problem, regular solution, existence, uniqueness

Introduction

The conjugation problems (generalized diffraction problems) for the equation

$$(-1)^{p+1}D_t^{2p}u - h(x)u_{xx} + c(x,t)u = f(x,t) \tag{$*$}$$

($D_t^{2p} = \frac{\partial^{2p}}{\partial t^{2p}}$, $p \geq 1$) in the case of a strictly positive function $h(x)$ continuous everywhere but possibly one interior point of the domain are studied in [1]. In this article we consider similar problems for the composite type equation

$$(-1)^{p+1}D_t^{2p}(u - g(x)u_{xx}) - h(x)u_{xx} + c(x,t)u = f(x,t) \tag{$**$}$$

in the case of a strictly positive function $g(x)$ whereas $h(x)$ is not necessarily positive. Note that the necessary boundary conditions for ($**$) are rather similar to those for ($*$) in [2–4].

The conjugation problems or the generalized diffraction problems are studied in mathematics and mathematical modeling for a long time (see, for instance, [5–17]). Recall the following: The problems with the conjugate (gluing) conditions arise naturally in the theory of mixed type equations; many articles are devoted to their study, for instance, [18–33] (actually, considerably many articles can be pointed out). At last, we note that, besides [1], the conjugation problems for nonclassical differential equations are studied in [34–37].

The general conjugation problem was proposed in [1]; but the conditions on the coefficients provide four special cases which were investigated. In this article we discuss special cases from the very beginning.

1. Statements of the Problems

Assume that Ω is the interval $(-1, 1)$ of the Ox-axis and Q is the rectangle $\Omega \times (0, T)$, $0 < T < +\infty$. Next, assume that $g(x)$, $h(x)$, $c(x,t)$, and $f(x,t)$ are given functions defined for $x \in \overline{\Omega}$ and $t \in [0, T]$, and $g(x)$ is strictly positive for $x \in \overline{\Omega}$,

a_i, b_i, c_i, and d_i, $i = \overline{1,4}$ are given reals. Put $D_t^k = \frac{\partial^k}{\partial t^k}$. Let L be a differential operator whose action on a given function $v(x,t)$ is defined as

$$Lv = (-1)^{p+1} D_t^{2p}(v - g(x))v_{xx} - h(x)v_{xx} + c(x,t)v$$

(here $p \geq 1$ is a positive integer).

Put $Q_1 = (-1,0) \times (0,T)$, $Q_2 = (0,1) \times (0,T)$, and $Q_0 = Q_1 \cup Q_2$.

Conjugation Problem I. *Find a solution $u(x,t)$ to the equation*

$$Lu = f(x,t) \tag{1}$$

in Q_0 satisfying the boundary conditions

$$D_t^k u(x,t)\big|_{t=0} = 0, \quad x \in (-1,0) \cup (0,1), \quad k = 0, \dots, p, \tag{2}$$

$$D_t^k u(x,t)\big|_{t=T} = 0, \quad x \in (-1,0) \cup (0,1), \quad k = 1, \dots, p-1 \tag{3}$$

(*for $p = 1$, this condition is absent*),

$$u(-1,t) = u(1,t) = 0, \quad t \in (0,T), \tag{4}$$

and the conjugate conditions

$$u(-0,t) = a_1 u_x(-0,t) + b_1 u_x(+0,t), \quad t \in (0,T), \tag{5}$$

$$u(+0,t) = c_1 u_x(-0,t) + d_1 u_x(+0,t), \quad t \in (0,T). \tag{6}$$

Conjugation Problem II. *Find a solution $u(x,t)$ to* (1) *in Q_0 satisfying the boundary conditions* (2)–(4), *and the conjugate conditions*

$$u(-0,t) = a_2 u(+0,t) + b_2 u_x(-0,t), \quad t \in (0,T), \tag{7}$$

$$u_x(+0,t) = c_2 u(+0,t) + d_2 u_x(-0,t), \quad t \in (0,T). \tag{8}$$

Conjugation Problem III. *Find a solution $u(x,t)$ to* (1) *in Q_0 satisfying the boundary conditions* (2)–(4) *and the conjugate conditions*

$$u(+0,t) = a_3 u(-0,t) + b_3 u_x(+0,t), \quad t \in (0,T), \tag{9}$$

$$u_x(-0,t) = c_3 u(-0,t) + d_3 u_x(+0,t), \quad t \in (0,T). \tag{10}$$

Conjugation Problem IV. *Find a solution $u(x,t)$ to* (1) *in Q_0 satisfying the boundary conditions* (2)–(4) *and the conjugate conditions*

$$u_x(-0,t) = a_4 u(-0,t) + b_4 u(+0,t), \quad t \in (0,T), \tag{11}$$

$$u_x(+0,t) = c_4 u(-0,t) + d_4 u(+0,t), \quad t \in (0,T). \tag{12}$$

Note that Problems II and III correspond to the classical diffraction problems, where the gluing conditions of a solution and its gradient are imposed.

Define the space in which we study the uniqueness and existence of solutions to Conjugation Problems I–IV. Namely, put

$$V_m = \big\{v(x,t) : \ v(x,t) \in W_{2,x,t}^{2,m}(Q_1), \ v(x,t) \in W_{2,x,t}^{2,m}(Q_2),$$
$$v_{xx}(x,t) \in W_{2,x,t}^{2,m}(Q_1), \ v_{xx}(x,t) \in W_{2,x,t}^{2,m}(Q_2)\big\}.$$

Endow V_m with the norm

$$\|v\|_{V_m} = \big(\|v\|^2_{W_{2,x,t}^{2,m}(Q_1)} + \|v_{xx}\|^2_{W_{2,x,t}^{2,m}(Q_1)} + \|v\|^2_{W_{2,x,t}^{2,m}(Q_2)} + \|v_{xx}\|^2_{W_{2,x,t}^{2,m}(Q_2)}\big)^{\frac{1}{2}}.$$

Obviously, in this case V_m becomes a Banach space.

2. Uniqueness of Solutions

In what follows, we assume that $g(-0)$, $g(+0)$, $h(-0)$, and $h(+0)$ are finite for $g(x)$ and $h(x)$. Put

$$h_1(x) = \begin{cases} h(x) & \text{for } x \in [-1,0), \\ h(-0) & \text{for } x = 0, \end{cases} \qquad h_2(x) = \begin{cases} h(x) & \text{for } x \in (0,1), \\ h(+0) & \text{for } x = 0, \end{cases}$$

$$g_1(x) = \begin{cases} g(x) & \text{for } x \in [-1,0), \\ g(-0) & \text{for } x = 0, \end{cases} \qquad g_2(x) = \begin{cases} g(x) & \text{for } x \in (0,1), \\ h(+0) & \text{for } x = 0, \end{cases}$$

$$\varphi_1(x) = \frac{h_1(x)}{g_1(x)}, \quad \varphi_2(x) = \frac{h_2(x)}{g_2(x)}.$$

Let $\varphi_1(x)$ and $\varphi_2(x)$ be representable as

$$\varphi_1(x) = \varphi_{10}(x) + \varphi_{11}(x), \quad \varphi_2(x) = \varphi_{20}(x) + \varphi_{21}(x).$$

Put

$$\bar{g}_{11} = \max_{-1\le x\le 0} g_1(x), \quad \bar{g}_{21} = \max_{0\le x\le 1} g_2(x), \quad C_1 = \min_{\overline{Q}_1} \frac{c(x,t)}{g_1(x)}, \quad C_2 = \min_{\overline{Q}_2} \frac{c(x,t)}{g_2(x)},$$

$$k_1 = \sup_{-1\le x\le 0} |\varphi_{11}(x)|, \quad k_2 = \sup_{0\le x\le 1} |\varphi_{21}(x)|,$$

$$m_1 = \sup_{-1\le x\le 0} |\varphi_1'(x)|, \quad m_2 = \sup_{0\le x\le 1} |\varphi_2'(x)|.$$

The functions $v(x,t)$ satisfying (2) and belonging to V_{2p} meet the inequalities

$$\int\limits_{Q_1} v^2\,dxdt \le M_0 \int\limits_{Q_1} \left(D_t^p v\right)^2 dxdt, \quad \int\limits_{Q_2} v^2\,dxdt \le M_0 \int\limits_{Q_2} \left(D_t^p v\right)^2 dxdt, \tag{13}$$

$$\int\limits_{Q_1} v_t^2\,dxdt \le M_1 \int\limits_{Q_1} \left(D_t^p v\right)^2 dxdt + M_2 \int\limits_{Q_1} v^2\,dxdt, \tag{14}$$

$$\int\limits_{Q_2} v_t^2\,dxdt \le N_1 \int\limits_{Q_2} \left(D_t^p v\right)^2 dxdt + N_2 \int\limits_{Q_2} v^2\,dxdt, \tag{15}$$

where M_0 is defined only by the number T, while M_1 and N_1 are either arbitrary positive numbers (in this case M_2 and N_2 are calculated through M_1 and N_1, respectively, and T) or M_1 and N_1 are defined by T ($M_2 = 0$ and $N_2 = 0$ in this case).

Let λ_0 be a number in $[T, +\infty)$. Put

$$A_1 = 2p - 1 - k_1 M_{-}(\lambda_0 - T) k_1 M_0 T - m_1 M_0 T,$$

$$A_2 = \frac{2p-1}{\bar{g}_{11}} - m_1 M_1 T, \quad A_3 = C_1 - \frac{m_1 M_2 T}{2},$$

$$B_1 = 2p - 1 - k_2 M_0 - (\lambda_0 - T) k_2 M_0 T - m_2 M_0 T,$$

$$B_2 = \frac{2p-1}{\bar{g}_{21}} - m_2 N_1 T, \quad B_3 = C_2 - \frac{m_2 N_2 T}{2}.$$

Theorem 1. *Assume that*

$$g_1(x) \in C^1([-1,0]), \quad g_2(x) \in C^1([0,1]), \quad 0 < \bar{g}_{01} \le g_1(x) \le \bar{g}_{11} \\ \text{for} \quad x \in [-1,0], \quad 0 < \bar{g}_{02} \le g_2(x) \le \bar{g}_{21} \quad \text{for} \quad x \in [0,1]; \tag{16}$$

$$h_1(x) \in C^1([-1,0]), \quad h_2(x) \in C^1([0,1]); \tag{17}$$

$$\varphi_{10}(x) \ge 0 \quad \text{for } x \in [-1,0], \quad \varphi_2(x) \ge 0 \quad \text{for } x \in [0,1]; \tag{18}$$

$$c(x,t) \in C^1(\overline{Q}), \quad c(x,t) \ge 0, \quad c_t(x,t) \le 0 \quad \text{for } (x,t) \in \overline{Q}; \tag{19}$$

$$a_1 \le 0, \; b_1 c_1 \le 0, \quad d_1 \ge 0, \quad a_1 d_1 - b_1 c_1 \le 0; \tag{20}$$

$$\varphi_1(0) \ge 0, \quad \varphi_2(0) \ge 0, \quad b_1[\varphi_1(0) - \varphi_2(0)] = 0; \quad \text{if } c_1 \ne 0, \\ \text{then the quadratic form} \quad -a_1\varphi_1(0)\xi^2 - 2b_1\varphi_2(0)\xi\eta - \frac{b_1 d_1 \varphi_2(0)}{c_1}\eta^2 \\ \text{is nonnegative definite for } (\xi,\eta) \in \mathbb{R}^2; \tag{21}$$

$$A_i \ge 0, \quad B_i \ge 0, \quad i = \overline{1,3}, \quad A_1^2 + A_2^2 + A_3^2 > 0, \quad B_1^2 + B_2^2 + B_3^2 > 0. \tag{22}$$

Then Conjugation Problem I *has at most one solution in* V_{2p}.

Proof. First, we assume that $b_1 \ne 0$, $c_1 \ne 0$, and $f(x,t)$ is identically zero in Q. Consider the equality

$$\int\limits_{Q_1} \frac{\lambda_0 - t}{g(x)} Lu \cdot u_t \, dxdt + \gamma \int\limits_{Q_2} \frac{\lambda_0 - t}{g(x)} Lu \cdot u_t \, dxdt = 0, \tag{23}$$

where γ coincides with $-\frac{b_1}{c_1}$, and λ_0 is as above. Integrating by parts and using (2)–(4), we conclude from (23) that

$$\frac{2p-1}{2} \int\limits_{Q_1} \frac{1}{g(x)} \big(D_t^p u\big)^2 \, dxdt + \frac{2p-1}{2} \int\limits_{Q_1} \big(D_t^p u_x\big)^2 \, dxdt \\
+ \frac{\gamma(2p-1)}{2} \int\limits_{Q_2} \frac{1}{g(x)} \big(D_t^p u\big)^2 \, dxdt + \frac{\gamma(2p-1)}{2} \int\limits_{Q_2} \big(D_t^p u_x\big)^2 \, dxdt \\
+ \frac{1}{2} \int\limits_{Q_1} \varphi_{10}(x) u_x^2 \, dxdt + \frac{\gamma}{2} \int\limits_{Q_2} \varphi_{20}(x) u_x^2 \, dxdt \\
+ \int\limits_{Q_1} \frac{c(x,t) - (\lambda_0 - t) c_t(x,t)}{g(x)} u^2 \, dxdt + \gamma \int\limits_{Q_2} \frac{c(x,t) - (\lambda_0 - t) c_t(x,t)}{g(x)} u^2 \, dxdt \\
+ \frac{\lambda_0 - T}{2} \int\limits_{-1}^{0} \frac{1}{g(x)} \big[D_t^p u(x,T)\big]^2 \, dx + \frac{\lambda_0 - T}{2} \int\limits_{-1}^{0} \big[D_t^p u_x(x,T)\big]^2 \, dx \\
+ \frac{\gamma(\lambda_0 - T)}{2} \int\limits_{0}^{1} \frac{1}{g(x)} \big[D_t^p u(x,T)\big]^2 \, dx + \frac{\gamma(\lambda_0 - T)}{2} \int\limits_{0}^{1} \big[D_t^p u_x(x,T)\big]^2 \, dx \\
+ \frac{\lambda_0 - T}{2} \int\limits_{-1}^{0} \frac{c(x,t)}{g(x)} u^2(x,T) \, dx + \frac{\gamma(\lambda_0 - T)}{2} \int\limits_{0}^{1} \frac{c(x,t)}{g(x)} u^2(x,T) \, dx$$

$$
\begin{aligned}
&+\int_0^T\Big\{-\frac{a_1}{2}\big[D_t^p u_x(-0,t)\big]^2+\gamma c_1 D_t^p u_x(+0,t)D_t^p u_x(-0,t)\\
&+\frac{\gamma d_1}{2}\big[D_t^p u_x(+0,t)\big]^2\Big\}\,dt-\frac{a_1(\lambda_0-T)}{2}\big[D_t^p u_x(-0,T)\big]^2\\
&+\gamma c_1(\lambda_0-T)D_t^p u_x(+0,T)D_t^p u_x(-0,T)+\frac{\gamma d_1(\lambda_0-T)}{2}\big[D_t^p u_x(+0,T)\big]^2\\
&+(p-1)\int_0^T\big\{-a_1\big[D_t^p u_x(-0,t)\big]^2+(\gamma c_1-b_1)D_t^p u_x(+0,t)D_t^p u_x(-0,t)\\
&+\gamma d_1\big[D_t^p u_x(+0,t)\big]^2\big\}\,dt+\frac{\lambda_0-T}{2}\int_{-1}^0\varphi_{10}(x)u_x^2(x,T)\,dx\\
&+\frac{\gamma(\lambda_0-T)}{2}\int_0^1\varphi_{20}(x)u_x^2(x,T)\,dx+\int_0^T\Big\{-\frac{a_1\varphi_1(0)}{2}u_x^2(-0,t)\\
&+\gamma c_1\varphi_2(0)u_x(+0,t)u_x(-0,t)+\frac{\gamma d_1\varphi_2(0)}{2}u_x^2(+0,t)\Big\}\,dt\\
&-\frac{a_1\varphi_1(0)}{2}u_x^2(-0,T)+\gamma c_1\varphi_2(0)u_x(+0,T)u_x(-0,T)+\frac{\gamma d_1\varphi_2(0)}{2}u_x^2(+0,T)\\
&=-\frac{1}{2}\int_{Q_1}\varphi_{11}(x)u_x^2\,dxdt-\frac{\gamma}{2}\int_{Q_2}\varphi_{21}(x)u_x^2\,dxdt-\int_{Q_1}(\lambda_0-t)\varphi_1'(x)u_xu_t\,dxdt\\
&-\gamma\int_{Q_2}(\lambda_0-t)\varphi_2'(x)u_xu_t\,dxdt-\frac{\lambda_0-T}{2}\int_{-1}^0\varphi_{11}(x)u_x^2(x,T)\,dx\\
&-\frac{\gamma(\lambda_0-T)}{2}\int_0^1\varphi_{21}(x)u_x^2(x,T)\,dx. \qquad (24)
\end{aligned}
$$

Taking account of (13)–(15), the elementary inequalities

$$
\int_{-1}^0 u_x^2(x,T)\,dx\le T\int_{Q_1}u_{xt}^2\,dxdt,\qquad \int_0^1 u_x^2(x,T)\,dx\le T\int_{Q_2}u_{xt}^2\,dxdt,
$$

and the Young inequality, we can estimate the right-hand side in (24) as follows:

$$
\begin{aligned}
&\Big[\frac{k_1M_0}{2}+\frac{(\lambda_0-T)k_1M_0T}{2}+\frac{m_1M_0T}{2}\Big]\int_{Q_1}\big[D_t^p u_x\big]^2\,dxdt\\
&+\frac{m_1M_1T}{2}\int_{Q_1}\big[D_t^p u\big]^2\,dxdt+\frac{m_1M_2T}{2}\int_{Q_1}u^2\,dxdt\\
&+\gamma\Big[\frac{k_2M_0}{2}+\frac{(\lambda_0-T)k_2M_0T}{2}+\frac{m_2M_0T}{2}\Big]\int_{Q_2}\big[D_t^p u_x\big]^2\,dxdt\\
&+\frac{\gamma m_2N_1T}{2}\int_{Q_2}\big[D_t^p u\big]^2\,dxdt+\frac{\gamma m_2N_2T}{2}\int_{Q_2}u^2\,dxdt.
\end{aligned}
$$

This estimate, (16)–(21), and (23) yield

$$\frac{A_1}{2}\int\limits_{Q_1}\left[D_t^p u_x\right]^2 dxdt+\frac{A_2}{2}\int\limits_{Q_1}\left[D_t^p u\right]^2 dxdt+A_3\int\limits_{Q_1}u^2\,dxdt$$

$$+\frac{\gamma B_1}{2}\int\limits_{Q_2}\left[D_t^p u_x\right]^2 dxdt+\frac{\gamma B_2}{2}\int\limits_{Q_2}\left[D_t^p u\right]^2 dxdt+\gamma B_3\int\limits_{Q_2}u^2\,dxdt\le 0. \tag{25}$$

This inequality and (22) imply that $u(x,t)\equiv 0$ in Q_1 and $u(x,t)\equiv 0$ in Q_2.

We assume now that $b_1=0$. In this case Problem I in Q_1 is a separate problem; analyzing the equality

$$\int\limits_{Q_1}\frac{\lambda_0-t}{g(x)}Lu\cdot u_t\,dxdt=0 \tag{26}$$

for a solution $u(x,t)$ and using the conditions of the theorem, we infer $u(x,t)\equiv 0$ in Q_1. Condition (6) gives rise to another problem in Q_2; analyzing the equality

$$\int\limits_{Q_2}\frac{\lambda_0-t}{g(x)}Lu\cdot u_t\,dxdt=0 \tag{27}$$

and involving the conditions of the theorem, we infer $u(x,t)\equiv 0$ in Q_2.

In the case of $c_1=0$, the arguments are similar.

The theorem is proven.

Theorem 2. *Assume conditions* (16)–(19), (22) *of Theorem* 1 *and the conditions*

$$b_2\le 0,\quad c_2\ge 0,\quad a_2d_2\ge 0; \tag{28}$$

$$\varphi_1(0)\ge 0,\quad \varphi_2(0)\ge 0,\quad a_2[\varphi_1(0)-\varphi_2(0)]=0. \tag{29}$$

Then Conjugation Problem II *has at most one solution in* V_{2p}.

The proof of Theorem 2 is rather similar to that of Theorem 1. Namely, we analyze (23) in which γ coincides with $\frac{a_2}{d_2}$ in the case of $a_2\ne 0$ and $d_2\ne 0$ or (26) or (27) in the case of $a_2=0$ or $d_2=0$.

Theorem 3. *Assume that conditions* (16)–(19), (22) *of Theorem* 1 *and the conditions*

$$b_3\ge 0,\quad c_3\le 0,\quad a_3d_3\ge 0; \tag{30}$$

$$\varphi_1(0)\ge 0,\quad \varphi_2(0)\ge 0,\quad d_3[\varphi_1(0)-\varphi_2(0)]=0 \tag{31}$$

are fulfilled. Then Conjugation Problem III *has at most one solution in* V_{2p}.

To demonstrate this theorem, we employ (23) where γ coincides with $\frac{a_3}{d_3}$.

Theorem 4. *Assume that conditions* (16)–(19), (22) *of Theorem* 1 *and the conditions*

$$a_4\le 0,\quad b_4c_4\le 0,\quad d_4\ge 0,\quad b_4c_4\le a_4d_4; \tag{32}$$

$$\varphi_1(0)\ge 0,\quad \varphi_2(0)\ge 0,\quad b_4[\varphi_1(0)-\varphi_2(0)]=0 \tag{33}$$

are fulfilled. Then Conjugation Problem IV *has at most one solution in* V_{2p}.

To justify Theorems 3 and 4, we employ the same equality (23) with numbers $\gamma=\frac{a_3}{d_3}$ and $-\frac{b_4}{c_4}$, respectively, or one of the equalities (26) and (27).

Remark. The conditions (28), (30), and (32) can be weakened if we assume additionally that

$$\varphi_1(x)\ge m_{01}>0\quad\text{for}\quad x\in[-1,0],\quad \varphi_2(x)\ge m_{02}>0\quad\text{for}\quad x\in[0,1]$$

(this situation is treated in [1]).

3. Existence of Solutions

To prove solvability of Conjugation Problems I–IV, as in [1], we apply the method of continuation in a parameter.

Following [1], we first examine Conjugation Problem II.

Theorem 5. *Assume* (16)–(19), (22), (28), *and* (29) *fulfilled. Then Conjugation Problem* II *is solvable in* V_{2p} *for every* $f(x,t)$ *in* $L_2(Q)$.

Proof. Let μ be a number in $[0,1]$. Examine the family of problems: *Find a solution* $u(x,t)$ *to the equation*

$$L_\mu u \equiv (-1)^{p+1} D_t^{2p}(u - g(x)u_{xx}) - \mu[h(x)u_{xx} + c(x,t)u] = f \tag{1_μ}$$

in Q *satisfying* (2)–(4) *and the conditions*

$$u(-0,t) = \mu[a_2 u(+0,t) + b_2 u_x(-0,t)], \quad t \in (0,T), \tag{7_μ}$$

$$u_x(+0,t) = \mu[c_2 u(+0,t) + d_2 u_x(-0,t)], \quad t \in (0,T). \tag{8_μ}$$

Note that this problem is solvable in V_{2p} for $\mu = 0$, since Problem (1_0), (2)–(4), (7_0), (8_0) decomposes into two independent problems for ordinary differential equations in the variables t and x whose solvability is obvious. By the method of continuation in a parameter [38], Problem (1_μ), (2)–(4), (7_μ), (8_μ) is solvable in V_{2p} for all μ in $[0,1]$ whenever solutions $u(x,t)$ to this problem satisfy an a priori estimate uniform (in μ)

$$\|u\|_{V_{2p}} \le R_0 \tag{34}$$

with a constant R_0 defined only by the coefficients of L, the number T, and the norm of $f(x,t)$ in $L_2(Q)$.

Demonstrate this estimate.

Assume that $a_2 \ne 0$ and $d_2 \ne 0$. Examine the equality

$$\int\limits_{Q_1} \frac{\lambda_0 - t}{g(x)} Lu \cdot u_t \,dxdt + \gamma \int\limits_{Q_2} \frac{\lambda_0 - t}{g(x)} Lu \cdot u_t \,dxdt$$
$$= \int\limits_{Q_1} \frac{\lambda_0 - t}{g(x)} f \cdot u_t \,dxdt + \gamma \int\limits_{Q_2} \frac{\lambda_0 - t}{g(x)} f \cdot u_t \,dxdt$$

(λ_0 and γ are those of the proof of Theorem 2).

Repeating the arguments of the analysis of (23) and using (2)–(4), (7_μ), (8_μ), the conditions of the theorem, and the Young inequality, we arrive at the first a priori estimate for a solution $u(x,t)$ to Problem (1_μ), (2)–(4), (7_μ), (8_μ) of the form

$$\int\limits_{Q_1} \left(D_t^p u_x\right)^2 dxdt + \int\limits_{Q_2} \left(D_t^p u_x\right)^2 dxdt \le R_1, \tag{35}$$

where the constant R_1 in this estimate is defined only by the coefficients of L, the number T, and the norm of $f(x,t)$ in $L_2(Q)$. At the next step we consider the equality

$$\int\limits_{Q_1} \frac{\lambda_0 - t}{g(x)} Lu \cdot u_{xxt} \,dxdt + \gamma \int\limits_{Q_2} \frac{\lambda_0 - t}{g(x)} Lu \cdot u_{xxt} \,dxdt$$
$$= \int\limits_{Q_1} \frac{\lambda_0 - t}{g(x)} f \cdot u_{xxt} \,dxdt + \gamma \int\limits_{Q_2} \frac{\lambda_0 - t}{g(x)} f \cdot u_{xxt} \,dxdt.$$

Integrating by parts and using (2)–(4), (7_μ), (8_μ), the conditions of the theorem, and the Young inequality we see that a solution $u(x,t)$ to Problem (1_μ), (2)–(4), (7_μ), (8_μ) satisfies the estimate

$$\int\limits_{Q_1} (D_t^p u_{xx})^2\,dxdt + \int\limits_{Q_2} (D_t^p u_{xx})^2\,dxdt \leq R_2 \tag{36}$$

with the constant R_2 defined by the coefficients of L, the number T, and the norm of $f(x,t)$ in $L_2(Q)$.

Analyzing the equality

$$\int\limits_{Q_1} Lu\cdot D_t^{2p}u_{xx}\,dxdt + \gamma\int\limits_{Q_2} Lu\cdot D_t^{2p}u_{xx}\,dxdt$$
$$= \int\limits_{Q_1} f\cdot D_t^{2p}u_{xx}\,dxdt + \gamma\int\limits_{Q_2} f\cdot D_t^{2p}u_{xx}\,dxdt,$$

with the use of the conditions of the theorem, (35), (36), and the Young inequality, we infer

$$\int\limits_{Q_1} \left(D_t^{2p}u_{xx}\right)^2 dxdt + \int\limits_{Q_2} \left(D_t^{2p}u_{xx}\right)^2 dxdt \leq R_3, \tag{37}$$

where the constant R_3 is defined by the coefficients of L, the number T, and the norm of $f(x,t)$ in $L_2(Q)$.

The estimates (35)–(37) imply (34). As is mentioned above, the validity of this estimate ensures the solvability of Problem (1_μ), (2)–(4), (7_μ), (8_μ) for all μ in $[0,1]$, in particular, for $\mu = 1$.

So, we proved the claim for $a_2 \neq 0$ and $d_2 \neq 0$. If $a_2 = 0$ or $d_2 = 0$ then in one of the rectangles Q_1 or Q_2 (or in Q_1 and Q_2) Conjugation Problem II turns into a usual initial-boundary value problem whose solvability results from the a priori estimate (34) and the method of continuation in a parameter (the derivation of (34) in this case is similar to that as before but first we prove it in one of the rectangles Q_1 or Q_2 and after that in the other).

The above arguments imply that, for $a_2 = 0$ or $d_2 = 0$, Conjugation Problem II is also solvable in V_{2p}.

The theorem is proven.

Theorem 6. *Let* (16)–(19), (21), (30), *and* (31) *hold. Then, for every* $f(x,t)$ *in* $L_2(Q)$, *Conjugation Problem* III *is solvable in* V_{2p}.

The proof of the theorem is similar to that of Theorem 5; namely, we involve a priori estimates and the method of continuation in a parameter.

Theorem 7. *Let* (16)–(22) *hold. Then, for every function* $f(x,t)$ *from* $L_2(Q)$, *Conjugation Problem* I *in solvable in* V_{2p}.

Theorem 8. *Let* (16)–(19), (22), (32), *and* (33) *hold. Then, for every function* $f(x,t)$ *from* $L_2(Q)$, *Conjugation Problem* IV *is solvable in* V_{2p}.

The proofs of Theorems 7 and 8 are similar to those of the corresponding theorems in [1]; namely, we employ Theorems 5 and 6 on solvability of Conjugation Problems II and III.

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A NUMERICAL SOLUTION OF THE INVERSE STEFAN PROBLEM BY INTRODUCING A DISTRIBUTED HEAT SOURCE

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Abstract. We consider the two-phase inverse Stefan problem of reconstructing the right-hand side of the heat equation as a function of time given its spatial distribution. We propose a new method for accounting for the heat of the phase passage by introducing a heat source distributed in a neighborhood of the phase transition boundary. An algorithm is constructed for computations based on transforming the original problem to a boundary value problem for the loaded heat equation and present examples of simulations.

Keywords: heat of the phase passage, distributed heat source, Stefan problem, boundary value problem for a loaded equation, inverse problem, reconstruction of the right-hand side of the heat equation, mesh problem, difference scheme

Introduction

The problems of reconstructing unknown heat sources from additional temperature measurements at separate points are important in the inverse problems of mathematical physics [1–3]. In many cases the unknown is the time dependence of the right-hand side. To approximately solve the problems of reconstructing the unknown right-hand side, we use various approaches that rest on regularization methods [4].

The series of studies [5, 6] bases the numerical algorithms for solving inverse problems approximately on transforming the original problem into a boundary value problem for loaded heat equations in which a special setup of calculations reduces the inverse problem to two direct problems. This method is used in [7] to numerically solve the simplest spatially one-dimensional single-phase inverse problem of reconstructing the variable intensivity of heat sources from their available spatial distribution.

In this article we consider the problem of reconstructing the time dependence of the right-hand side of a parabolic equation when the distribution in space is available. This linear inverse problem belongs to the class of ill-posed problems of mathematical physics in the classical sense under some special assumptions on the points of additional measurements: the source must act at the points of observation [5]. Basing on the method of [7], we construct a computational algorithm for approximately solving the spatially one-dimensional two-phase inverse Stefan problem with a new approach to accounting for the heat of the phase transition: we introduce a heat source distributed in the neighborhood of the phase transition boundary [8, 9].

1. Statement of the Problem

The distribution of temperature and position of the phase transition boundary is determined by the solution to the system

$$c_1 \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_1 \frac{\partial T}{\partial x}\right) + f(x,t), \quad 0 < x < \xi(t), \tag{1.1}$$

$$c_2 \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_2 \frac{\partial T}{\partial x}\right) + f(x,t), \quad \xi(t) < x < l. \tag{1.2}$$

The following conditions hold on the phase transition boundary:

$$T = T_*, \quad x = \xi(t), \tag{1.3}$$

$$\lambda_2 \frac{\partial T}{\partial x} - \lambda_1 \frac{\partial T}{\partial x} = L \frac{d\xi}{dt}. \tag{1.4}$$

For (1.1)–(1.4) we impose boundary conditions of the third kind:

$$\lambda_1 \frac{\partial T}{\partial x} = \alpha(T - T_C), \quad x = 0, \tag{1.5}$$

$$\frac{\partial T}{\partial x} = 0, \quad x = l. \tag{1.6}$$

Moreover, we impose the initial condition

$$T(x,0) = T_C, \quad 0 \le x \le l. \tag{1.7}$$

In (1.1)–(1.7) we use the following notation: c is the spatial heat capacity, λ is the heat conductivity coefficient, T_* and T_C are the temperatures of the phase transition and the surrounding medium, L is the latent heat of crystallization (melting), α is the heat transfer coefficient, and $x = \xi(t)$ is the equation of the phase transition boundary. The indices 1 and 2 refer to the phases with $T > T_*$ and $T < T_*$.

The direct problem is stated as (1.1)–(1.7). In this article we consider the inverse problem in which, apart from $T(x,t)$, the right-hand side $f(x,t)$ of (1.1), (1.2) is unknown.

Assume that we can express $f(x,t)$ as

$$f(x,t) = \tilde{c}\eta(t)\psi(x), \tag{1.8}$$

where $\psi(x)$ is a known function and $\eta(t)$ is the time dependence of the source. We reconstruct this dependence from the additional observation of $T(x,t)$ at some interior point $0 < x^* < l$:

$$T(x^*,t) = \varphi(t). \tag{1.9}$$

We solve the problem of reconstructing the time dependence of the right-hand side of the parabolic equation with known spatial distribution while imposing the following restrictions:

(1) $\psi(x^*) \neq 0$;

(2) $\psi(x)$ is sufficiently smooth ($\psi \in C^2[0,1]$);

(3) to simplify exposition, $\psi(x) = 0$ on the boundary of the computational domain.

The first assumption merits particular attention: a source is acting at the point x^* of observation. Precisely this makes the identification problem well-posed, meaning a continuous dependence of the solution on the initial data, right-hand side, and measurements at the interior point.

2. The Method of Introducing a Distributed Heat Source

A number of methods are available for numerical solutions to the Stefan problem (1.1)–(1.7). The most widely used is the method of [10, 11], where the enthalpy function is used to reduce the Stefan problem to a boundary value problem for the heat equation with discontinuous coefficients. Constructing a difference scheme of end-to-end computation for equations of this type, we replace the Dirac delta function by a delta-like function on the interval $(T_* - \Delta, T_* + \Delta)$. Moreover, we approximate the enthalpy function by various continuous functions on this interval. Approximation by the unit Heaviside function instead of the enthalpy function is considered in [12]. These methods assume that the phase transition of crystallization starts at a certain temperature above the crystallization temperature.

To numerically solve the inverse Stefan problem, we propose in this article to modify the approach to accounting for the heat of the phase transition. Our approach describes the real process of heat release on the phase transition boundary more precisely thanks to the introduction of a heat source [8] distributed in the neighborhood $(T_* - \Delta, T_*]$ (toward the forming phase). To this end, we introduce a piecewise continuous nonnegative function satisfying the following conditions:

(1) $v(T)$ is defined on the entire range of temperatures, is nonzero on the interval $(T_* - \Delta, T_*)$, and vanishes identically outside it;

(2) $v(T_*) = 1$;

(3) $\frac{\partial v}{\partial T} < 0$ for $T \in (T_* - \Delta, T_*)$.

Proposition. *If $v(T)$ satisfies these conditions then we can replace* (1.1), (1.2), *and* (1.4) *by the single equation* (2.1) *on the whole domain $0 < x < l$:*

$$c\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + L\frac{\partial v}{\partial t} + f(x,t), \tag{2.1}$$

where $c = c_1$ and $\lambda = \lambda_1$ for $T > T_*$, while $c = c_2$ and $\lambda = \lambda_2$ for $T < T_*$.

Proof. The proof of the proposition, by using the method of [10], appears in [8, 9]: in [8] in the case of a one-dimensional domain and vanishing right-hand sides of (1.1) and (1.2) ($f(x,t) = 0$); in [9] in the case of a two-dimensional domain and $f(x,t) \neq 0$.

As functions of temperature, the coefficients of (2.1) have discontinuities at $T = T_*$ and are undefined at the point. Combine the last term on the right-hand side with left-hand side. Then the effective heat capacity $c - L\frac{dv}{dT}$ has discontinuities at $T = T_*$ and $T = T_* + \Delta$. Using this, we rearrange (2.1) as

$$\tilde{c}\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\tilde{\lambda} \frac{\partial T}{\partial x}\right) + f, \tag{2.2}$$

where the coefficients are defined on the whole range of temperatures as

$$\tilde{c} = \begin{cases} c_2 & \text{if } T \le T_* - \Delta, \\ c_2 - L\frac{dv}{dT} & \text{if } T_* - \Delta < T \le T_*, \\ 0.5(c_1 + c_2) - L\frac{dv}{dT} & \text{if } T = T_*, \\ c_1 & \text{if } T \ge T_*, \end{cases} \tag{2.3}$$

$$\tilde{\lambda} = \begin{cases} \lambda_1 & \text{if } T \le T_* - \Delta, \\ (\lambda_1 + \lambda_2)/2 - (\lambda_2 + \lambda_1)(T - T_*)/2\Delta & \text{if } T_* - \Delta \le T \le T_* + \Delta, \\ \lambda_2 & \text{if } T > T_* + \Delta. \end{cases} \tag{2.4}$$

We approximate the heat conductivity coefficient by a continuous function. We can also extend it by analogy with the heat capacity coefficient. Since the temperature field is most sensitive to the value of the heat capacity coefficient, we must extend it to correspond most precisely to the real phase transition process. This explains our choice of the function $\tilde{c}$ in the above form.

Thus, the direct problem (1.1)–(1.7) reduces to (2.2) with the boundary conditions (1.5) and (1.6), as well as the initial condition (1.7).

Choosing the function $v(T)$, we must take into account the direction of the process: does the phase transition result from lowering or raising the temperature? We must choose $v(T)$ to be nonvanishing on the domain of forming phase.

Our numerical solution of the inverse problem under consideration uses the method of [5, 7], which amounts to reducing the inverse problem to a boundary value problem for a loaded equation and constructing a difference scheme and a nonlocal mesh problem to implement it.

3. Reducing the Inverse Problem to a Boundary Value Problem for the Loaded Equation

Seek a solution to the inverse problem as

$$T(x,t) = \theta(t)\psi(x) + \omega(x,t), \tag{3.1}$$

where

$$\theta(t) = \int_0^t \eta(s)\,ds. \tag{3.2}$$

Inserting (1.8), (1.9), and (3.2) into (2.2) yields the equation for $\omega(x,t)$:

$$\tilde{c}\frac{\partial \omega}{\partial t} = \frac{\partial}{\partial x}\left(\tilde{\lambda}\frac{\partial \omega}{\partial x}\right) + \theta(t)\frac{\partial}{\partial x}\left(\tilde{\lambda}\frac{\partial \psi}{\partial x}\right). \tag{3.3}$$

Taking (3.2) into account, we see that (1.9) leads to the expression

$$\theta(t) = \frac{1}{\psi(x^*)}(\varphi(t) - \omega(x^*,t)) \tag{3.4}$$

for the unknown $\theta(t)$. Inserting (3.4) into (3.3) yields the required loaded parabolic equation

$$\tilde{c}\frac{\partial \omega}{\partial t} = \frac{\partial}{\partial x}\left(\tilde{\lambda}\frac{\partial \omega}{\partial x}\right) + \frac{1}{\psi(x^*)}\Big(\varphi(t) - \omega(x^*,t)\Big)\frac{\partial}{\partial x}\left(\tilde{\lambda}\frac{\partial \psi}{\partial x}\right). \tag{3.5}$$

By the assumptions on the right-hand side along the boundary, the boundary condition is

$$\tilde{\lambda}\frac{\partial w(0,t)}{\partial x} = \alpha(w(0,t) + T_C), \quad \frac{\partial w(l,t)}{\partial x} = 0, \quad 0 \le t \le t_0. \tag{3.6}$$

It follows from (3.2) that the auxiliary function $\theta(t)$ satisfies

$$\theta(0) = 0, \tag{3.7}$$

which enables us to use the initial condition

$$w(x,0) = T_C, \quad 0 < x < l. \tag{3.8}$$

Therefore, we state the inverse problem (2.2), (1.5)–(1.7) as a boundary value problem for the loaded equations (3.5)–(3.8) with the expression (3.2), (3.4) for the unknown time dependence of the source.

4. The Difference Scheme

Take the uniform mesh $\overline{\omega}$ with meshsize h along x. Denote by $x_i = ih$ for $i = 0, 1, \dots, N$ with $Nh = l$ the nodes of this mesh and assume that $v = v_i = v(x_i)$. For simplicity, assume that the point $x = x^*$ of observation coincides with the interior node corresponding to the index $i = k$.

To pass from one temporal layer $t_j = j\tau$, where $j = 0, 1, \dots, j_0$ and $\tau > 0$, to the next layer t_{j+1}, we use a purely implicit difference scheme for (3.5). At the interior nodes of the spatial mesh we have

$$\tilde{c}\frac{\omega^{j+1} - \omega^j}{\tau} = \big(a\omega_{\bar{x}}^{j+1}\big)_x + \frac{1}{\psi_k}\big(\varphi^{j+1} - \omega_k^{j+1}\big)(a\psi_{\bar{x}})_x. \tag{4.1}$$

For problems with sufficiently smooth coefficient $\tilde{\lambda}$ put $a_i = \tilde{\lambda} \cdot 0,5(x_i + x_{i-1})$. Approximating (3.6) and (3.8), we obtain

$$\tilde{\lambda}\frac{\omega_1^{j+1} - \omega_0^{j+1}}{h} = \alpha\big(\omega_0^{j+1} - T_C\big),$$

$$\frac{\omega_N^{j+1} - \omega_{N-1}^{j+1}}{h} = 0, \quad j = 0, 1, \dots, j_0 - 1, \tag{4.2}$$

$$\omega_i^0 = T_C, \quad i = 1, 2, \dots, N-1. \tag{4.3}$$

From the solution to the difference problem (4.1)–(4.3), in accordance with (3.4) define

$$\theta^{j+1} = \frac{1}{\psi_k}\big(\varphi^{j+1} - \omega_k^{j+1}\big), \quad j = 0, 1, \dots, j_0 - 1, \tag{4.4}$$

complementing these relations by the condition $\theta^0 = 0$ (see (3.7)). Taking (3.2) into account, for the required time dependence of the right-hand side we use the simplest numerical differentiation procedure:

$$\eta^{j+1} = \frac{\theta^{j+1} - \theta^j}{\tau}, \quad j = 0, 1, \dots, j_0 - 1. \tag{4.5}$$

It is necessary to dwell particularly on the questions of solving the mesh problem in order to implement the implicit scheme under consideration.

5. The Nonlocal Mesh Problem and Software Implementation

Even though the mesh problem on the new temporal layer is nonstandard (nonlocal), the implementation of the scheme (4.1)–(4.3) encounters no particular difficulties. Following the method of [5, 7], write down (4.1) at the interior nodes as

$$\tilde{c}\frac{\omega_i^{j+1}}{\tau} - \big(a\omega_{\bar{x}}^{j+1}\big)_{x,i} + \frac{1}{\psi_k}(a\psi_{\bar{x}})_{x,i}\omega_k^{j+1} = g_i^j \tag{5.1}$$

with a prescribed right-hand side g_i^j and boundary conditions (4.2). Seek the solution to (4.2), (5.1) as

$$\omega_i^{j+1} = y_i + \omega_k^{j+1} z_i, \quad i = 0, 1, \dots, N. \tag{5.2}$$

Inserting (5.2) into (5.1) enables us to state for the auxiliary functions y_i and z_i the mesh problems

$$\tilde{c}\frac{y_i}{\tau} - (ay_{\bar{x}})_{x,i} = g_i^j, \quad i = 1, 2, \ldots, N-1, \tag{5.3}$$

$$\tilde{\lambda}\frac{y_1 - y_0}{h} = \alpha(y_0 - T_C), \quad \tilde{\lambda}\frac{y_N - y_{N-1}}{h} = 0, \tag{5.4}$$

$$\tilde{c}\frac{z_i}{\tau} - (az_{\bar{x}})_{x,i} + \frac{1}{\psi_k}(a\psi_{\bar{x}})_{x,i} = 0, \quad i = 1, 2, \ldots, N-1, \tag{5.5}$$

$$\tilde{\lambda}\frac{z_1 - z_0}{h} = \alpha z_0, \quad -\tilde{\lambda}\frac{z_N - z_{N-1}}{h} = 0. \tag{5.6}$$

Then, taking (5.2) into account, we find

$$\omega_k^{j+1} = \frac{y_k}{1 - z_k}. \tag{5.7}$$

The algorithm is guaranteed to be correct since the denominator in (5.7) never vanishes. By the maximum principle for difference schemes, for the mesh problem (5.5), (5.6) we establish the a priori estimate

$$\max_{0 \le i \le N} |z_i| \le \tau \max_{0 < i < N} \left|\frac{1}{\psi_k}\right| (a\psi_{\bar{x}})_{x,i}.$$

Thus, $|z_i| < 1$ for sufficiently small $\tau = O(1)$; that is, we must use small time steps.

We obtain the algorithm for solving our inverse problem numerically. At each time step, successively perform the following calculations:

(1) Given the previous distribution of temperature, choose the parameter Δ to smooth out the discontinuous coefficients $\tilde{c}$ and $\tilde{\lambda}$ so that the smoothing interval $(T_* - \Delta, T_*]$ includes the values of temperature at least at two neighboring nodes on the opposite sides of the phase transition boundary. If none of the values of temperature at the nodes belongs to this interval then we will determine the temperature field while ignoring the heat release of the phase transition.

(2) Introduce the distributed source function $v(T)$ satisfying the hypotheses of the proposition and use it to determine the values of the discontinuous coefficients $\tilde{c}$ and $\tilde{\lambda}$ in accordance with (2.3) and (2.4).

(3) Solve the auxiliary mesh problems: (5.3), (5.4) for y and (5.5), (5.6) for z.

(4) Successively determine the values of ω_k^{j+1} using (5.7), ω_i^{j+1} using (5.2), and θ^{j+1} using (4.4).

(5) Reconstruct the unknown time dependence η^{j+1} of the right-hand side using (4.5).

(6) Use (3.1) to find the temperature field for the current temporal layer and assign it to the previous distribution of temperature.

(7) Iterate (1)–(6) until the condition

$$|T_\delta(x,t) - T(x,t)| < \delta$$

becomes true, where δ is the accuracy threshold, and $T(x,t)$ is the solution to the problem. Then proceed to the next time layer.

We used this algorithm to run simulations. In the framework of the concept of quasireal experiments we consider the direct problem (2.2), (1.5)–(1.7) with a prescribed right-hand side (1.8), where

$$\eta(t) = \begin{cases} t, & \text{if } 0 < t < 0.6, \\ 0, & \text{if } t \ge 0.6, \end{cases} \qquad \psi(x) = \sin\left(\frac{\pi x}{l}\right).$$

We consider two variants for $v(T)$:
(1) $v(T) = 1 + \frac{T_* - T}{\Delta}$;
(2) $v(T) = \exp(0.69(T_* - T + \Delta)/\Delta) - 1$.

We solve this problem with the following values of the parameters: $c_1 = 2814$ kJ/m^3·K, $\lambda_1 = 6.3$ kJ/m·h·K, $c_2 = 2016$ kJ/m^3·K, $\lambda_2 = 8.4$ kJ/m·h·K, $T_* = 0$°C, $T_C = -25$°C, $L = 0.4175 \cdot 10^6$ kJ/m^3, $\alpha = 83.5$ kJ/m^2·h·K, $l = 0.1$ m, $N = 100$, $k = 40$, and $\tau = 0.0125$.

To solve the inverse problem (3.1)–(3.8), we determine the additional observation function $\varphi(t)$ from the values of $T(x,t)$ at the point $x^* = 0.6$ found as the solution to the direct problem.

To check that the algorithm is correct, we compare the computed results for the inverse problem with the results of the direct problem.

Table 1

	Nodes	0	5	10	20	30	40	50
1	T_1, °C	-0.210	1.481	3.007	5.611	7.445	8.481	8.805
2	T_2, °C	-0.262	1.589	3.178	5.701	7.376	8.411	8.814
3	T_1, °C	-0.210	1.584	3.070	5.632	7.450	8.482	8.806
4	T_3, °C	-0.261	1.593	3.184	5.711	7.385	8.416	8.816
5	T_1, °C	-0.312	1.404	2.961	5.598	7.441	8.480	8.805
6	T_4, °C	-0.262	1.587	3.174	5.694	7.368	8.403	8.799

Table 1 shows the computed results at the nodes for $j_0 = 50$ ($t = 0.625$ hours). In the first two rows we compare the values of temperature which are obtained by solving the direct problem (T_1) and the inverse problem (T_2) for the first form of the function $v(T)$; in rows 3 and 4, the results of the direct problem (T_1) and the inverse problem (T_3) for the second form of the function $v(T)$; in rows 5 and 6, the results of the direct problem (T_1) and the inverse problem (T_4) for the traditional smoothing-out method.

It is clear from the table that the values of temperature obtained using the traditional smoothing-out method (rows 5 and 6) are below the values obtained by the proposed method of introducing a distributed heat source. The reason is the chosen value of the spatial heat capacity, which in the traditional smoothing-out method is greater than the actual value.

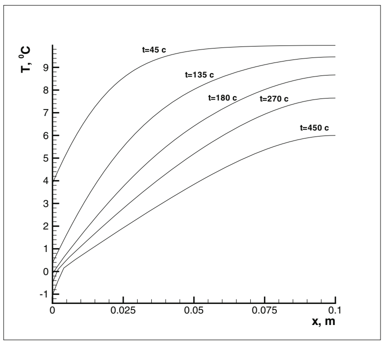

Fig. 1. Distribution of temperature ($\delta = 0.5$)

Fig. 1 depicts the distribution of the temperature field obtained by solving the inverse problem with $v(T) = 1 + \frac{T_* - T}{\Delta}$ at various times, where δ is the accuracy threshold.

The data computed for various accuracy thresholds illustrates the well-posedness of the inverse problem under consideration and the proposed method of accounting for the heat of the phase transition. Fig. 2 depicts on the left the solution for $\delta = 0.75$, and on the right for $\delta = 0.25$. As the accuracy threshold decreases, the solution is reconstructed more precisely.

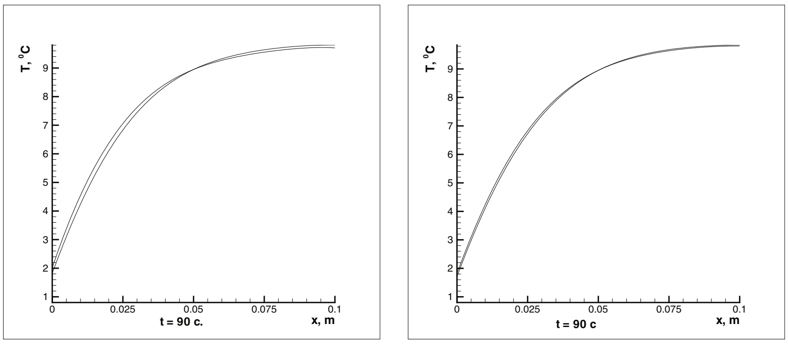

Fig. 2. Distribution of temperature for various accuracy thresholds

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STUDYING CONTACT PARABOLIC BOUNDARY VALUE PROBLEMS IN HÖLDER SPACES

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Abstract. We examine forward-backward parabolic equations of the second order with gluing conditions containing functions of variables $t \in [0, T]$ with the use of the theory of singular integral equations. Solvability is established of boundary value problems in Hölder spaces. We also demonstrate that the Hölder classes of solutions depend on a noninteger Hölder exponent and the signs of coefficients occurring in the gluing conditions at the ends of the interval $[0, T]$ provided that some necessary and sufficient conditions on the input data of the problem are fulfilled.

Keywords: forward-backward parabolic equation, gluing condition, well-posedness, Hölder space, singular integral equation

In [1] there is proposed a unified approach to constructing conjugate models for different physical processes; among them heat transfer in inhomogeneous media (problems of diffraction type), interaction of filtration and channel flows of fluids (filtration in a borehole), reverse flow in a boundary layer after the separation point, etc. In particular, the solvability is established in [2–4] of boundary value problems for forward-backward parabolic equations in Hölder spaces in domains with the interface modeling opposite cocurrent flows. In this article we examine the general case with an interface of two media, namely, the forward-backward parabolic equations with gluing conditions containing coefficients depending on time.

In a domain $Q = \Omega \times (0, T)$, $\Omega \equiv \mathbb{R}$, we consider the equation

$$g(x)u_t = u_{xx}, \quad g(x) = \operatorname{sgn} x. \tag{1}$$

A solution to (1) is sought in the Hölder space $H^{p,p/2}_{x\ t}(Q^{\pm})$, $Q^{\pm} = \mathbb{R}^{\pm} \times (0, T)$, $p = 2l + \gamma$, $0 < \gamma < 1$, and it satisfies the initial conditions

$$u(x, 0) = \varphi_1(x),\ x > 0, \quad u(x, T) = \varphi_2(x),\ x < 0, \tag{2}$$

and the gluing conditions

$$u(-0, t) = u(+0, t), \quad a(t) \cdot u_x(-0, t) = u_x(+0, t), \tag{3}$$

where $l \geq 1$ is an integer, $\varphi_1(x)$, $\varphi_2(x)$, and $a(t)$ are given functions defined for $x \in \mathbb{R}$, $t \in [0, T]$.

For convenience, we replace (1) with the system of equations

$$u^1_t = u^1_{xx}, \quad -u^2_t = u^2_{xx} \tag{4}$$

in Q^+. The initial conditions and the gluing conditions are rewritten as

$$u^1(x, 0) = \varphi_1(x), \quad u^2(x, T) = \varphi_2(x), \quad x > 0, \tag{5}$$

$$u^1(0, t) = u^2(0, t), \quad u^1_x(0, t) + a(t) \cdot u^2_x(0, t) = 0. \tag{6}$$

Assume that $\varphi_i(x) \in H^p(\mathbb{R})$, $i = 1, 2$. In this case

$$\begin{aligned}\omega_1(x,t) &= \frac{1}{2\sqrt{\pi t}} \int\limits_{\mathbb{R}} \exp\Big(- \frac{(x-\xi)^2}{4t}\Big) \varphi_1(\xi)\, d\xi, \\ \omega_2(x,t) &= \frac{1}{2\sqrt{\pi(T-t)}} \int\limits_{\mathbb{R}} \exp\Big(- \frac{(x-\xi)^2}{4(T-t)}\Big) \varphi_2(\xi)\, d\xi\end{aligned} \tag{7}$$

are solutions to (4) satisfying (5) in $\mathbb{R}$. In our method we employ the following integral representations for solutions of the system of equations (4):

$$\begin{aligned}u^1(x,t) &= \frac{1}{\sqrt{\pi}} \int\limits_0^t \exp\Big(- \frac{x^2}{4(t-\tau)}\Big) (t-\tau)^{-\frac{1}{2}} \alpha(\tau)\, d\tau + \omega_1(x,t), \\ u^2(x,t) &= \frac{1}{\sqrt{\pi}} \int\limits_t^T \exp\Big(- \frac{x^2}{4(\tau-t)}\Big) (\tau-t)^{-\frac{1}{2}} \beta(\tau)\, d\tau + \omega_2(x,t).\end{aligned} \tag{8}$$

The functions, defined by (8), meet (5) and (4).

In accord with [5, 6], the functions $u^k(x,t)$, $k = 1, 2$, belong to $H^{p,p/2}_{x\ t}(Q^+)$ whenever the unknown densities $\alpha(t)$ and $\beta(t)$ lie in $H^{(p-1)/2}(0,T)$; in this case we have that

$$\alpha^{(s)}(0) = \beta^{(s)}(T) = 0, \quad s = 0, \dots, l-1. \tag{9}$$

Using (6), we arrive at the system of equations for $\alpha(t)$ and $\beta(t)$:

$$\begin{cases} \int\limits_0^t \frac{\alpha(\tau)}{(t-\tau)^{\frac{1}{2}}}\, d\tau = \int\limits_t^T \frac{\beta(\tau)}{(\tau-t)^{\frac{1}{2}}}\, d\tau + \Phi_0(t), \\ \alpha(t) + a(t)\beta(t) = \Phi_1(t), \end{cases} \tag{10}$$

where

$$\Phi_0(t) = \sqrt{\pi}(\omega_2(0,t) - \omega_1(0,t)), \quad \Phi_1(t) = a(t) \cdot \omega_{2x}(0,t) + \omega_{1x}(0,t).$$

If we invert the first equation in (10) with the use of the celebrated Abel inversion formulas, we find that

$$\begin{cases} \alpha(t) - \frac{1}{\pi} \int\limits_0^T (\frac{\tau}{t})^{1/2} \frac{\beta(\tau)}{\tau - t}\, d\tau = \frac{d}{dt} \int\limits_0^t \frac{\Phi_0(\tau)}{(t-\tau)^{\frac{1}{2}}}\, d\tau, \\ \alpha(t) + a(t)\beta(t) = \Phi_1(t). \end{cases} \tag{11}$$

Put $F_1^s(t) = \Phi_1^{(s)}(t) - \Phi_1^{(s)}(0)$,

$$F_0^s(t) = \int\limits_0^t \frac{\Phi_0^{(s+1)}(\tau) - \Phi_0^{(s+1)}(0)}{(t-\tau)^{\frac{1}{2}}}\, d\tau, \quad s = 0, \dots, l-1.$$

As is easily seen, $F_0^{l-1}(t)$ and $F_1^{l-1}(t)$ belong to the Hölder space with exponent $(1+\gamma)/2$; in this case $F_0^{l-1}(t) = F_1^{l-1}(t) = O(t^{(1+\gamma)/2})$ for small t.

If we assume that $\alpha(t)$ and $\beta(t)$ belong to $H^{(p-1)/2}(0,T)$ then (11) yields

$$\int\limits_0^T \frac{\beta(\tau)}{\tau^{1/2}}\, d\tau = -\pi \Phi_0(0), \quad a(0)\beta(0) = \Phi_1(0). \tag{12}$$

Assuming (12), we can rewrite (11) as

$$\begin{cases} \alpha(t) - \frac{1}{\pi}\int\limits_0^T \left(\frac{t}{\tau}\right)^{1/2} \frac{\beta(\tau)}{\tau - t}\, d\tau = 2\Phi_0'(0)t^{1/2} + F_0^0(t), \\ \alpha(t) + a(t)\beta(t) - a(0)\beta(0) = F_1^0(t). \end{cases} \tag{13}$$

Introduce the new unknowns $\bar{\beta}(t) = \beta(t) - \beta(0)\frac{T-t}{T}$ in (13). In this case (13) is representable as

$$\begin{cases} \alpha(t) - \frac{1}{\pi}\int\limits_0^T \left(\frac{t}{\tau}\right)^{1/2} \frac{\bar{\beta}(\tau)}{\tau - t}\, d\tau \\ \qquad = 2\Phi_0'(0)t^{1/2} + F_0^0(t) - \frac{4}{\pi}\beta(0)F\big(-\frac{1}{2}, 1, \frac{3}{2}; \frac{t}{T}\big)\left(\frac{t}{T}\right)^{\frac{1}{2}}, \\ \alpha(t) + a(t)\bar{\beta}(t) = a(t)\beta(0)\frac{t}{T} - \beta(0)[a(t) - a(0)] + F_1^0(t). \end{cases} \tag{14}$$

If $l > 1$ then differentiate (14) to see that

$$\begin{cases} \alpha'(t) - \frac{1}{2\pi}\left(t^{-\frac{1}{2}}\int\limits_0^T \frac{\bar{\beta}(\tau)}{\tau^{1/2}(\tau - t)}\, d\tau + 2t^{\frac{1}{2}}\frac{d}{dt}\int\limits_0^T \frac{\bar{\beta}(\tau)}{\tau^{1/2}(\tau - t)}\, d\tau\right) \\ = -\frac{2}{\pi}\beta(0)F\big(-\frac{1}{2}, 1, \frac{1}{2}; \frac{t}{T}\big)(tT)^{-\frac{1}{2}} + \Phi_0'(0)t^{-\frac{1}{2}} + 2\Phi_0''(0)t^{\frac{1}{2}} + F_0^1(t), \\ \alpha'(t) + a(t)\bar{\beta}'(t) = a'(t)\big[-\bar{\beta}(t) + \beta(0)\frac{t}{T} - \beta(0)\big] \\ \qquad\qquad + a(t)\beta(0)\frac{1}{T} + \Phi_1'(0) + F_1^1(t). \end{cases} \tag{15}$$

Using this system, we infer

$$\int\limits_0^T \frac{\bar{\beta}(\tau)}{\tau^{3/2}}\, d\tau = \frac{4}{\sqrt{T}}\beta(0) - 2\pi\Phi_0'(0), \tag{16}$$
$$a(0)\beta'(0) + a'(0)\beta(0) = (a(t)\beta(t))'|_{t=0} = \Phi_1'(0).$$

Now (15) under the conditions (16) for $\beta(T) = 0$ is rewritten as

$$\begin{cases} \alpha'(t) - \frac{1}{\pi}\int\limits_0^T \left(\frac{t}{\tau}\right)^{1/2} \frac{\beta'(\tau)}{\tau - t}\, d\tau = 2\Phi_0''(0)t^{\frac{1}{2}} + F_0^1(t), \\ \alpha'(t) + (a(t)\beta(t))' - (a(t)\beta(t))'|_{t=0} = F_1^1(t). \end{cases} \tag{17}$$

Note that (17) are of the same form as (13). It is easy to check that under the conditions

$$\begin{cases} \int\limits_0^T \frac{\beta^{(s)}(\tau) - \beta^{(s)}(0)\frac{T-\tau}{T}}{\tau^{3/2}}\, d\tau = \frac{4}{\sqrt{T}}\beta^{(s)}(0) - 2\pi\Phi_0^{(s+1)}(0), \\ (a(t)\beta(t))^{(s+1)}|_{t=0} = \Phi_1^{(s+1)}(0), \quad \beta^{(s)}(T) = 0, \end{cases} \qquad s = 1, \dots, l-2, \tag{18}$$

we obtain the system of equations

$$\begin{cases} \alpha^{(l-1)}(t) - \frac{1}{\pi}\int\limits_0^T \left(\frac{t}{\tau}\right)^{1/2} \frac{\beta^{(l-1)}(\tau)}{\tau - t}\, d\tau = 2\Phi_0^{(l)}(0)t^{\frac{1}{2}} + F_0^{l-1}(t), \\ \alpha^{(l-1)}(t) + (a(t)\beta(t))^{(l-1)} - (a(t)\beta(t))^{(l-1)}|_{t=0} = F_1^{l-1}(t). \end{cases} \tag{19}$$

Introduce the new unknown function $\tilde{\beta}^{(l-1)}(t) = \beta^{(l-1)}(t) - \beta^{(l-1)}(0)\frac{T-t}{T}$. In this case (19) is rewritten as

$$\begin{cases} \alpha^{(l-1)}(t) - \frac{1}{\pi}\int\limits_0^T \left(\frac{t}{\tau}\right)^{1/2} \frac{\tilde{\beta}^{(l-1)}(\tau)}{\tau - t}\, d\tau = 2\Phi_0^{(l)}(0)t^{1/2} \\ \qquad + F_0^{l-1}(t) - \frac{4}{\pi}\beta^{(l-1)}(0)F\big(-\frac{1}{2}, 1, \frac{3}{2}; \frac{t}{T}\big)\left(\frac{t}{T}\right)^{\frac{1}{2}}, \\ \alpha^{(l-1)}(t) + a(t)\tilde{\beta}^{(l-1)}(t) = -P(t) + P(0) - a(t)\beta^{(l-1)}(0)\frac{t}{T} \\ \qquad + \beta^{(l-1)}(0)(a(0) - a(t)) + F_1^{l-1}(t), \end{cases} \tag{20}$$

where

$$P(t) = \sum_{k=1}^{l-1} C_{l-1}^{k} a^{(k)}(t)\beta^{(l-1-k)}(t).$$

Since $\alpha^{(l-1)}(t)$ and $\beta^{(l-1)}(t) \in H^{(1+\gamma)/2}(0,T)$, it is necessary that

$$\int_0^T \frac{\tilde{\beta}^{(l-1)}(\tau)}{\tau^{3/2}}\, d\tau = \frac{4}{\sqrt{T}}\beta^{(l-1)}(0) - 2\pi\Phi_0^{(l)}(0). \tag{21}$$

In this case we arrive at the system of equations

$$\begin{cases} \alpha^{(l-1)}(t) - \frac{1}{\pi}\int\limits_0^T \left(\frac{t}{\tau}\right)^{3/2} \frac{\tilde{\beta}^{(l-1)}(\tau)}{\tau - t}\, d\tau = \overline{F}_0^{\,l-1}(t), \\ \alpha^{(l-1)}(t) + a(t)\tilde{\beta}^{(l-1)}(t) = \overline{F}_1^{l-1}(t), \end{cases} \tag{22}$$

where

$$\overline{F}_0^{l-1}(t) = F_0^{l-1}(t) - \frac{4}{\pi}\beta^{(l-1)}(0)\left[F\left(-\frac{1}{2}, 1, \frac{3}{2}; \frac{t}{T}\right) - 1\right]\left(\frac{t}{T}\right)^{\frac{1}{2}},$$

$$\overline{F}_1^{l}(t) = F_1^{\,l-1}(t) - P(t) + P(0) - a(t)\beta^{(l-1)}(0)\frac{t}{T}$$

belong to $H^{(1+\gamma)/2}(0,T)$ and

$$\overline{F}_0^{\,l-1}(t) = \overline{F}_1^{\,l-1}(t) = O(t^{(1+\gamma)/2})$$

for small t.

Excluding $\alpha^{(l-1)}(t)$ in (22), we derive the singular equation

$$a(t)\tilde{\beta}^{(l-1)}(t) + \frac{1}{\pi}\int_0^T \left(\frac{t}{\tau}\right)^{3/2} \frac{\tilde{\beta}^{(l-1)}(\tau)}{\tau - t}\, d\tau = Q(t) \tag{23}$$

for $\tilde{\beta}^{(l-1)}(t)$, where $Q(t) = \overline{F}_1^{l-1}(t) - \overline{F}_0^{\,l-1}(t)$.

We treat the singular integral equation (23) as an equation for $\beta_0(t) = \tilde{\beta}^{(l-1)}(t)\, t^{-\frac{3}{2}}$. Find solutions $\beta_0(t)$ unbounded for $t = 0$ (with singularity less than 1) and bounded for $t = T$. To this end, we introduce the piecewise holomorphic function (see [7, 8])

$$\Psi(z) = \frac{1}{2\pi i}\int_0^T \frac{\beta_0(\tau)}{\tau - z}\, d\tau.$$

By Sokhotskiĭ–Plemelj formulas, (23) is equivalent to the Riemann problem

$$\begin{gathered} \Psi^+(t) = \frac{a(t) - i}{a(t) + i}\Psi^-(t) + \frac{Q(t)}{t^{\frac{3}{2}}(a(t) + i)}, \quad t \in (0,T), \\ \Psi^+(t) = \Psi^-(t), \quad t \in (-\infty, 0) \cup (0, +\infty), \end{gathered} \tag{24}$$

under the additional condition $\Psi(\infty) = 0$. Note that $G(t) = \frac{a(t)-i}{a(t)+i}$ and

$$\log G(t) = \begin{cases} 2i\arg(a(t) - i) = -2\pi i\theta, & a(t) > 0, \\ 2i\arg(|a(t)| + i) = 2\pi i\theta, & a(t) < 0, \end{cases}$$

where $\theta(t) = \frac{1}{\pi}\tan^{-1}\frac{1}{|a(t)|}$.

At the endpoints of the segment of integration $[0,T]$ we have the representation

$$\frac{1}{2\pi i}\int\limits_0^T \frac{\log G(\tau)}{\tau - z}\,d\tau = -\frac{\log G(0)}{2\pi i}\log z + \gamma_0(z),$$

$$\frac{1}{2\pi i}\int\limits_0^T \frac{\log G(\tau)}{\tau - z}\,d\tau = \frac{\log G(T)}{2\pi i}\log(z-T) + \gamma_T(z),$$

where $\gamma_0(z)$ and $\gamma_T(z)$ are bounded in some neighborhood of the endpoints of $[0,T]$.

We have the following four different cases: 1) $a(0)$ and $a(T)$ are positive; 2) $a(0)$ and $a(T)$ are negative; 3) $a(0)$ is positive and $a(T)$ is negative; 4) $a(0)$ is negative and $a(T)$ is positive.

In cases 1 and 2 we assume that $a(t)$ is of the same sign for $t \in [0,T]$ and in cases 3 and 4 that $a(t)$ changes sigh only at one point $t_0 \in [0,T]$.

Put $\theta_1 = \theta(0)$ and $\theta_2 = \theta(T)$. In the above class, the canonical function is as follows: in case 1

$$\chi(z) = z^{-1}(z-T)\exp\bigg(-\int\limits_0^T \frac{\theta(\tau)}{\tau - z}\,d\tau\bigg) = z^{-1+\theta_1}(z-T)^{1-\theta_2}\omega_1(z), \quad \varkappa = 0,$$

in case 2

$$\chi(z) = \exp\bigg(\int\limits_0^T \frac{\theta(\tau)}{\tau - z}\,d\tau\bigg) = z^{-\theta_1}(z-T)^{\theta_2}\omega_2(z), \quad \varkappa = 0,$$

in case 3

$$\begin{aligned}\chi(z) &= z^{-1}(z-t_0)\exp\bigg(-\int\limits_0^{t_0} \frac{\theta(\tau)}{\tau - z}\,d\tau + \int\limits_{t_0}^T \frac{\theta(\tau)}{\tau - z}\,d\tau\bigg)\\ &= z^{-1+\theta_1}(z-T)^{\theta_2}\omega_3(z), \quad \varkappa = 0,\end{aligned}$$

and in case 4

$$\chi(z) = (z-T)\exp\bigg(\int\limits_0^{t_0} \frac{\theta(\tau)}{\tau - z}\,d\tau - \int\limits_{t_0}^T \frac{\theta(\tau)}{\tau - z}\,d\tau\bigg) = z^{-\theta_1}(z-T)^{1-\theta_2}\omega_4(z), \quad \varkappa = -1.$$

Note that $\omega_k(z) = \exp(\gamma_0(z))$ near the point $z = 0$ and $\omega_k(z) = \exp(\gamma_T(z))$ near the point $z = T$.

By the general theory [7, 8], each solution to (24) is of the form

$$\Psi(z) = \chi(z)\Phi(z) \tag{25}$$

and in case 4 ($\varkappa = -1$) some solution exists under the additional condition

$$\int\limits_0^T \frac{Q(\tau)}{\tau^{\frac{3}{2}}\chi(\tau)}\,d\tau = 0, \tag{26}$$

where

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_0^T \frac{Q(\tau)\,d\tau}{\tau^{\frac{3}{2}}\chi(\tau)(\tau - z)}.$$

In this case a solution to (23) is representable as

$$\tilde{\beta}^{(l-1)}(t) = t^{\frac{3}{2}}(\Psi^+(t) - \Psi^-(t)) = \frac{a(t)Q(t)}{1+a^2(t)} - \frac{t^{3/2}\chi(t)}{\pi(1+a^2(t))}\int_0^T \frac{Q(\tau)\,d\tau}{\tau^{3/2}\chi(\tau)(\tau-t)}; \quad (27)$$

i.e.,

$$\tilde{\beta}^{(l-1)}(t) = \frac{a(t)Q(t)}{1+a^2(t)} + \frac{t^{1/2+\theta_1}(T-t)^{1-\theta_2}\omega_1(t)}{\pi(1+a^2(t))}\int_0^T \frac{Q(\tau)\,d\tau}{\tau^{1/2+\theta_1}(T-\tau)^{1-\theta_2}\omega_1(\tau)(\tau-t)} \quad (28)$$

in case 1,

$$\tilde{\beta}^{(l-1)}(t) = \frac{a(t)Q(t)}{1+a^2(t)} + \frac{t^{3/2-\theta_1}(T-t)^{\theta_2}\omega_2(t)}{\pi(1+a^2(t))}\int_0^T \frac{Q(\tau)\,d\tau}{\tau^{3/2-\theta_1}(T-\tau)^{\theta_2}\omega_2(\tau)(\tau-t)} \quad (29)$$

in case 2,

$$\tilde{\beta}^{(l-1)}(t) = \frac{a(t)Q(t)}{1+a^2(t)} + \frac{t^{1/2+\theta_1}(T-t)^{\theta_2}\omega_3(t)}{\pi(1+a^2(t))}\int_0^T \frac{Q(\tau)\,d\tau}{\tau^{1/2+\theta_1}(T-\tau)^{\theta_2}\omega_3(\tau)(\tau-t)} \quad (30)$$

in case 3, and

$$\tilde{\beta}^{(l-1)}(t) = \frac{a(t)Q(t)}{1+a^2(t)} + \frac{t^{3/2-\theta_1}(T-t)^{1-\theta_2}(t-t_0)\omega_4(t)}{\pi(1+a^2(t))}$$
$$\times \int_0^T \frac{Q(\tau)\,d\tau}{\tau^{3/2-\theta_1}(T-\tau)^{1-\theta_2}(\tau-t_0)\omega_4(\tau)(\tau-t)} \quad (31)$$

in case 4. Since $Q(t)$ belongs to $H^{(1+\gamma)/2}(0,T)$, the functions $\tilde{\beta}^{(l-1)}(t)$ of the formulas (28)–(31) satisfy the Hölder condition with exponent $\frac{1+\gamma}{2}$ at all points of the contour $(0,T)$. Examine their behavior at the endpoints of the contour. By the formula describing the behavior of the Cauchy-type integral at the endpoints of the integration contour [8, p. 76], we can easily check that $\tilde{\beta}^{(l-1)}(0) = \tilde{\beta}^{(l-1)}(T) = 0$.

Next, to complete the study of their behavior at the endpoints of the contour, we apply the Muskheleshvili–Tersenov lemma [8, pp. 82–86; 2, pp. 14–17]. This lemma in case 1 implies that if $\theta_1+\theta_2 \geq \frac{1}{2}$ then $\tilde{\beta}^{(l-1)}(t)$ in (28) satisfies the Hölder condition with exponent $\frac{1+\gamma}{2}$ for $0<\gamma<1-2\theta_2$, with exponent $1-\theta_2$ for $1-2\theta_2<\gamma<1$, and with exponent $1-\theta_2-\varepsilon$ for $\gamma = 1-2\theta_2$. Moreover, if $\theta_1+\theta_2<\frac{1}{2}$ then $\tilde{\beta}^{(l-1)}(t)$ satisfies the Hölder condition with exponent $\frac{1+\gamma}{2}$ for $0<\gamma<2\theta_1$, with exponent $\frac{1}{2}+\theta_1$ for $2\theta_1<\gamma<1$, and with exponent $\frac{1}{2}+\theta_1-\varepsilon$ for $\gamma=2\theta_1$.

In cases 2 and 3 the inequality $\theta_2 < \frac{1+\gamma}{2}$ in (29) and (30) imply that $\tilde{\beta}^{(l-1)}(t)$ satisfies the Hölder condition with exponent θ_2. If we additionally assume that

$$\int_0^T \frac{Q(\tau)}{\tau^{\frac{3}{2}}(T-\tau)\chi(\tau)}\,d\tau = 0, \quad (32)$$

then $\tilde{\beta}^{(l-1)}(t)$ satisfies the Hölder condition with exponent $\frac{1+\gamma}{2}$ for all $0<\gamma<1$.

In case 4 the function $\tilde{\beta}^{(l-1)}(t)$ in (31) satisfies the Hölder condition with exponent $\frac{1+\gamma}{2}$ for $0 < \gamma < 1 - 2\theta_2$, with exponent $1 - \theta_2$ for $1 - 2\theta_2 < \gamma < 1$, and with exponent $1 - \theta_2 - \varepsilon$ for $\gamma = 1 - 2\theta_2$.

Thus, under the conditions (12), (16), (18), and (21) of the form

$$\begin{cases} \int\limits_0^T \frac{\beta(\tau)}{\tau^{1/2}}\, d\tau = -\pi\Phi_0(0), \\ \int\limits_0^T \frac{\beta^{(s)}(\tau)-\beta^{(s)}(0)\frac{T-\tau}{T}}{\tau^{3/2}}\, d\tau = \frac{4}{\sqrt{T}}\beta^{(s)}(0) - 2\pi\Phi_0^{(s+1)}(0), \\ s = 0, 1, \dots, l-1, \quad \beta^{(k)}(T) = 0, \quad k = 0, 1, \dots, l-2, \end{cases} \tag{33}$$

we obtain a function $\beta(t)$ from the Hölder space such that

$$(a(t)\beta(t))^{(s)}|_{t=0} = \Phi_1^{(s)}(0), \quad s = 0, 1, \dots, l-1, \quad \beta^{(l-1)}(T) = 0. \tag{34}$$

The values of $\beta^{(s)}(0)$ in (33) are uniquely defined by (34) and the values of $\beta^{(s)}(t)$ with the use of the Taylor formula

$$\beta^{(s)}(t) = \sum_{k=s}^{l-2} \frac{\beta^{(k)}(0)}{(k-s)!} t^{k-s} + \frac{1}{(l-2-s)!} \int\limits_0^t (t-\tau)^{l-2-s} \beta^{(l-1)}(\tau)\, d\tau, \ s = 0, 1, \dots, l-2.$$

In this case the conditions $\beta^{(k)}(T) = 0$ hold for $k = 0, 1, \dots, l-2$ if and only if

$$0 = \sum_{k=s}^{l-2} \frac{\beta^{(k)}(0)}{(k-s)!} T^{k-s} + \frac{1}{(l-2-s)!} \int\limits_0^T (T-\tau)^{l-2-s} \beta^{(l-1)}(\tau)\, d\tau, \quad s = 0, 1, \dots, l-2. \tag{35}$$

Inserting the above functions $\beta^{(s)}(t)$ in the first l conditions in (33), we infer

$$\begin{cases} \frac{1}{(l-2)!} \int\limits_0^T \tau^{l-3/2}\, d\tau \int\limits_0^1 (1-\sigma)^{l-2} \beta^{(l-1)}(\sigma\tau)\, d\sigma \\ \qquad\qquad = -\sum\limits_{k=0}^{l-2} \frac{\beta^{(k)}(0) T^{k+1/2}}{k!(k-1/2)} - \pi\Phi_0(0), \\ \frac{1}{(l-2-s)!} \int\limits_0^T \tau^{l-s-5/2}\, d\tau \int\limits_0^1 (1-\sigma)^{l-2-s} \beta^{(l-1)}(\sigma\tau)\, d\sigma \\ \qquad\qquad = -\sum\limits_{k=s}^{l-2} \frac{\beta^{(k)}(0) T^{k-s-1/2}}{(k-s)!(k-s-1/2)} - 2\pi\Phi_0^{(s+1)}(0), \quad s = 0, \dots, l-1. \end{cases} \tag{36}$$

Note that $\beta^{(l-1)}(t)$ can be determined from (27).

Therefore, we have proved the following theorems:

Theorem 1. *Assume that $\varphi_1, \varphi_2 \in H^p$, $p = 2l + \gamma$, $a(t) \in C^{\,l-1}([0,T])$, and $a(t) > 0$ for $t \in [0,T]$. Then, under the $2l$ conditions* (35) *and* (36), *there exists a unique solution to* (1), *satisfying* (2) *and* (3) *from the space*

1) $H_{x\ t}^{p,p/2}$ *if* $0 < \gamma < \min\{2\theta_1, 1 - 2\theta_2\}$;

2) $H_{x\ t}^{q,q/2}$, $q = 2l + \min\{2\theta_1, 1 - 2\theta_2\}$ *if* $\min\{2\theta_1, 1 - 2\theta_2\} < \gamma < 1$;

3) $H_{x\ \ t}^{q-\varepsilon,(q-\varepsilon)/2}$ *if* $\gamma = \min\{2\theta_1, 1 - 2\theta_2\}$, *where the positive constant ε is arbitrarily small.*

Theorem 2. *Assume that* $\varphi_1, \varphi_2 \in H^p$, $p = 2l + \gamma$, $a(t) \in C^{l-1}([0,T])$, $a(0) < 0$, $a(T) > 0$, *and* $a(t)$ *changes sign at one point. Then under the* $2l+1$ *conditions* (26), (35), *and* (36), *there exists a solution to* (1) *satisfying* (2) *and* (3) *from the space*

1) $H^{p,p/2}_{x\ t}$ *if* $0 < \gamma < 1 - 2\theta_2$;
2) $H^{q,q/2}_{x\ t}$, $q = 2l + 1 - 2\theta_2$, *if* $1 - 2\theta_2 < \gamma < 1$;
3) $H^{q-\varepsilon,(q-\varepsilon)/2}_{x\ \ t}$ *if* $\gamma = 1 - 2\theta_2$, *where the positive constant* ε *is arbitrarily small.*

Theorem 3. *Assume that* $\varphi_1, \varphi_2 \in H^p$, $p = 2l + \gamma$, $a(t) \in C^{l-1}([0,T])$ *and* $a(t) < 0$ *for* $t \in [0,T]$ *or* $a(0) > 0$, $a(T) < 0$, *and* $a(t)$ *changes sign at one point. Then under the conditions* (32), (35), (36) *(thus we have* $2l+1$ *conditions) there exists a solution to* (1) *satisfying* (2) *and* (3) *from* $H^{p,p/2}_{x\ t}$.

Remark 1. Under the conditions (35) and (36) ($2l$), a solution to (1)–(3) obtained in Theorem 3 belongs to the wider space $H^{p-1,(p-1)/2}_{x\ \ \ t}(Q^{\pm})$.

Remark 2. Solutions to (1)–(3) in Theorems 1–3 depend on the index $\varkappa$ of the Riemann problem (24) provided that $a(t)$ changes sign at an arbitrary number of points $t \in [0,T]$.

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THE EQUILIBRIUM PROBLEM FOR A VISCOELASTIC BODY WITH A THIN RIGID INCLUSION

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Abstract. We consider the equilibrium problem for a two-dimensional viscoelastic body with a thin rigid inclusion. The differential statement of the problem involves an integral condition accounting for the action of external forces on the rigid-hand part. We give an equivalent statement with variational inequality and use it to establish the unique solvability of the original problem. The additional properties of the solutions enable us to simplify the interpretation of the integral condition.

Keywords: viscoelastic body, thin rigid inclusion, variational method, quasistatic problem

Consider a two-dimensional viscoelastic body occupying in its natural undeformed state some domain $\Omega \subset \mathbb{R}^2$ with smooth boundary Γ and denote by $u = (u_1, u_2)$ the displacement of the points of the body.

Introduce the relations among the components of small deformation and stress tensors as

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad \sigma_{ij} = a_{ijkl}\varepsilon_{kl}(u), \quad i, j = 1, 2.$$

Here and henceforth we assume summation over repeated indices. The coefficients a_{ijkl} for $i, j, k, l = 1, 2$ are the components of a positive definite elasticity tensor enjoying the symmetry properties

$$a_{ijkl} = a_{jikl} = a_{klij},$$
$$a_{ijkl}\xi_{kl}\xi_{ij} \geq c_0|\xi|^2, \quad \xi_{ij} = \xi_{ji}, \quad c_0 = \text{const} > 0.$$

In order to state the quasistatic problem for the equations describing a viscoelastic medium, put

$$w(t, x) = u(t, x) + \int_0^t u(\tau, x)\, d\tau, \quad t \in (0, T). \tag{1}$$

Inserting (1) into $\varepsilon_{ij}(w(t, x))$, we obtain the relations for $\sigma_{ij}(t, x)$:

$$\sigma_{ij}(t, x) = a_{ijkl}(x)\varepsilon_{kl}(w(t, x)) = a_{ijkl}(x)\varepsilon_{kl}(u(t, x)) + \int_0^t a_{ijkl}(x)\varepsilon_{kl}(u(\tau, x))\, d\tau.$$

Therefore, at $x \in \Omega$ we have

$$\sigma_{ij}(t) = a_{ijkl}\varepsilon_{kl}(t) + \int_0^t a_{ijkl}\varepsilon_{kl}(\tau)\, d\tau, \quad t \in (0, T).$$

These equations correspond to the law $\dot{\sigma} = A\dot{\varepsilon} + A\varepsilon$ characterizing the viscoelastic state of the body, where $\dot{v}$ stands for differentiation with respect to time.

We use (1) in the equilibrium equations as well:

$$-\frac{\partial \sigma_{ij}(t,x)}{\partial x_j} = f_i(t,x), \quad i = 1,2,$$

where $f = (f_1, f_2)$ is the vector of external forces, while we find $\sigma_{ij}(t,x)$ from the formulas above.

Therefore, in contrast to the equilibrium equations used together with Hooke's law (the elastic state), in our problem we cannot calculate the components of deformation and stress tensors locally with respect to t since they depend on the full history of forces.

Some quasistationary boundary value problems for equations using relations similar to (1) were studied in [1–4].

The viscoelastic body under consideration has a thin rigid inclusion whose form is determined by a curve $\gamma \subset \Omega$ which is smooth nonclosed and has no self-intersections. Denote by $\nu = (\nu_1, \nu_2)$ the unit normal vector to γ and put $\Omega_\gamma = \Omega \setminus \bar{\gamma}$.

Suppose that we extend γ to intersect Γ so that Ω_γ is divided in two subdomains Ω_1 and Ω_2 with Lipschitz boundaries; furthermore, $\operatorname{mes}(\Gamma \cap \partial\Omega_i) \neq 0$ for $i = 1,2$.

In the framework of this model, the concept of *rigid inclusion* is described as follows: We introduce the space of infinitesimal rigid displacements $R(\gamma)$ (cp. [5]):

$$R(\gamma) = \{\rho = (\rho_1, \rho_2) \mid \rho(x) = Bx + C, \ x \in \gamma\},$$

where $B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$ and $C = (c^1, c^2)$ with $b, c^1, c^2 = \text{const}$, and the space

$$R_\gamma = \{\rho \in L^2(0,T; R(\gamma)) \mid \rho(t,x) = B(t)x + C(t) \text{ on } \gamma \times (0,T)\},$$

where $B(t)$ is the skew-symmetric matrix whose entries are functions independent of x: $B(t) = \begin{pmatrix} 0 & b(t) \\ -b(t) & 0 \end{pmatrix}$ and $C(t) = (c^1(t), c^2(t))$.

Say that a viscoelastic body has a *thin rigid inclusion* whenever the functions u on $\gamma \times (0,T)$ coincide with some element of R_γ:

$$u = \rho^0 \quad \text{on } \gamma \times (0,T), \quad \rho^0 \in R_\gamma.$$

Consider the differential statement of the equilibrium problem for a two-dimensional viscoelastic body with a thin rigid inclusion without delamination.

In the cylinder $Q = \Omega \times (0,T)$ find the functions u such that $u = \rho^0$ on $\gamma \times (0,T)$, with $\rho^0 \in R_\gamma$, and simultaneously in the cylinder $Q_\gamma = \Omega_\gamma \times (0,T)$ find σ_{ij}, for $i,j = 1,2$, satisfying

$$-\frac{\partial \sigma_{ij}(t,x)}{\partial x_j} = f_i(t,x), \ i = 1,2, \quad \text{on } Q_\gamma, \tag{2}$$

$$\sigma_{ij}(t,x) = a_{ijkl}(x)\varepsilon_{kl}(w(t,x)), \ i,j = 1,2, \quad \text{on } Q, \tag{3}$$

$$u(t,x) = 0 \quad \text{on } \Gamma \times (0,T), \tag{4}$$

$$\int_\gamma [\sigma_{ij}(t,x)\nu_j]\bar{\rho}_i(x)\,d\gamma = 0, \quad \bar{\rho} \in R(\gamma), \ \text{for almost all } t \in (0,T). \tag{5}$$

Here (2) amounts to the equilibrium equations for a prescribed external load f, and (3) describes the viscoelastic state. In these equations the components of σ and ε depend on w, that is, involve integrals of the form $\int_0^t u(\tau,x)d\tau$. The boundary condition (4) determines the clamping of the body along its boundary. Condition (5) accounts for the vector of surface forces on the curve of rigid inclusion.

The solvability of problems concerning rigid inclusions in elastic bodies, as well as the properties of their solutions, were studied in [5–10].

Consider the bilinear form

$$b(u,\bar u)=\int_\Omega a_{ijkl}\varepsilon_{kl}(u)\varepsilon_{ij}(\bar u)\,d\Omega$$

and the function space

$$H_\gamma=\{v=(v_1,v_2)\in L^2(0,T;H^1(\Omega))\mid v=0 \\ \text{on } \Gamma\times(0,T),\ v=\rho \text{ on } \gamma\times(0,T),\ \rho\in R_\gamma\}.$$

Denote by V the dual space of H_γ. Introduce the linear operator $\Lambda: H_\gamma\to V$ with

$$(\Lambda u,\bar u)=\int_0^T b(w,\bar u)\,dt,\quad \bar u\in H_\gamma.$$

Observe that in our notation

$$b(w,\bar u)=b\Big(u+\int_0^t u\,d\tau,\ \bar u\Big)=\int_\Omega a_{ijkl}(x)\varepsilon_{kl}\Big(u(t,x)+\int_0^t u(\tau,x)\,d\tau\Big)\varepsilon_{ij}(\bar u(t,x))\,d\Omega.$$

Theorem. *If $f(t,x)\in H^1(0,T;L^2(\Omega))$ and $a_{ijkl}(x)\in L^\infty(\Omega)$ for $i,j,k,l=1,2$ then problem (2)–(5) has a unique solution $u(t,x)\in H_\gamma$ and $\sigma_{ij}(t,x)\in L^2(Q_\gamma)$ satisfying $u_t(t,x)\in L^2(0,T;H^1(\Omega))$.*

To prove this theorem, we firstly establish a lemma on the existence of the unique solution to the problem with the operator Λ. Then we verify that this problem is an equivalent statement of (2)–(5). The unique solvability of our boundary value problem would follow. Note that [11–16] describe variational methods and their applications to elasticity and viscoelasticity.

Lemma 1. *The following problem*

$$u\in H_\gamma,\quad (\Lambda u,v)=\int_0^T\int_\Omega fv\,d\Omega\,dt,\quad v\in H_\gamma \tag{6}$$

has the unique solution $u(t,x)$.

Proof. First of all, observe that Korn's inequality [17]

$$\int_\Omega \varepsilon_{ij}(v)\varepsilon_{ij}(v)\,d\Omega\ge c_1\|v\|^2_{H^1_0(\Omega)},\quad v\in H^1_0(\Omega),$$

with a constant c_1 independent of v, yields

$$b(u,u)\ge c_2\|u\|^2_{H^1_0(\Omega)},\quad u\in H^1_0(\Omega). \tag{7}$$

Calculating

$$(\Lambda u,u)=\int_0^T b\,(w,u)\,dt=\int_0^T b\Big(u+\int_0^t u\ d\tau,u\Big)\,dt=\int_0^T b(u,u)\,dt+\frac12 b\Big(\int_0^T u\,dt,\int_0^T u\,dt\Big),$$

in view of (7) we obtain

$$(\Lambda u, u) \geq \|u\|^2_{H_\gamma}. \tag{8}$$

Consequently,

$$\frac{(\Lambda u, u)}{\|u\|_{H_\gamma}} \to +\infty, \quad \|u\|_{H_\gamma} \to +\infty;$$

i.e., Λ is a coercive operator. Since Λ is also monotone and continuous, we conclude that Λ is pseudomonotone. This implies [16] that (6) has a solution. Since the operator is strictly monotone, the solution is unique.

Let us now deduce an additional property of solutions to (6), namely, the existence of the derivative $u_t(t, x)$ in Ω. This will enable us to consider (6) on the cross-sections of the cylinder Q for fixed $t \in (0, T)$. Integration over τ from 0 to the current time t persists in the statement on the cross-sections.

Lemma 2. *If $f \in H^1(0, T; L^2(\Omega))$ then the derivative $u_t \in L^2(0, T; H^1(\Omega))$ of the solution to* (6) *exists.*

Proof. Expand (6) as

$$\int_0^T \int_\Omega a_{ijkl}\varepsilon_{kl}(w(t))\varepsilon_{ij}(v(t))\, d\Omega\, dt = \int_0^T \int_\Omega f(t)\, v(t)\, d\Omega\, dt, \quad v \in H_\gamma. \tag{9}$$

For convenience, we suppress the dependence of functions on x in what follows.

Put

$$H^{1,0}_\gamma(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma,\ v = \rho \text{ on } \gamma,\ \rho \in R(\gamma)\}.$$

Take $\alpha > 0$ and consider the function

$$v(\theta) = \begin{cases} \bar{v} - u(t), & \theta \in (t - \alpha, t + \alpha), \\ 0, & \theta \notin (t - \alpha, t + \alpha), \end{cases}$$

where $\bar{v} \in H^{1,0}_\gamma(\Omega)$ is a fixed element. Inserting $v(\theta)$ into (9) and dividing the result by 2α, we obtain

$$\frac{1}{2\alpha} \int_{t-\alpha}^{t+\alpha} \int_\Omega a_{ijkl}\varepsilon_{kl}(w(t))\varepsilon_{ij}(\bar{v} - u(t))\, d\Omega\, dt = \frac{1}{2\alpha} \int_{t-\alpha}^{t+\alpha} \int_\Omega f(t)(\bar{v} - u(t))\, d\Omega\, dt.$$

Hence,

$$\int_\Omega a_{ijkl}\varepsilon_{kl}(w(t))\varepsilon_{ij}(\bar{v} - u(t))\, d\Omega = \int_\Omega f(t)(\bar{v} - u(t))\, d\Omega \quad \text{for almost all } \ t \in (0, T) \tag{10}$$

as $\alpha \to 0$. Therefore,

$$b(w(t), \bar{v} - u(t)) = \int_\Omega f(t)(\bar{v} - u(t))\, d\Omega. \tag{11}$$

Suppose that $\bar{v} = u(t + h)$. Then

$$b(w(t), u(t + h) - u(t)) = \int_\Omega f(t)(u(t + h) - u(t))\, d\Omega. \tag{12}$$

Now examine (11) at $t+h$, taking $u(t)$ as $\bar{v}$. We obtain

$$b(w(t+h), u(t)-u(t+h)) = \int\limits_{\Omega} f(t+h)(u(t)-u(t+h))\,d\Omega. \tag{13}$$

Put

$$d_h v(t) = \frac{v(t+h)-v(t)}{h}, \quad d_h^{\tau} v(t) = \frac{1}{h}\int\limits_{t}^{t+h} v(\tau)d\tau, \quad h>0.$$

Adding up (12) and (13), we have

$$b\big(d_h u(t) + d_h^{\tau} u(t), d_h u(t)\big) = \int\limits_{\Omega} d_h f(t)\, d_h u(t)\, d\Omega.$$

Hence,

$$b(d_h u(t), d_h u(t)) = \int\limits_{\Omega} d_h f(t)\, d_h u(t)\, d\Omega - b\big(d_h^{\tau} u(t), d_h u(t)\big). \tag{14}$$

Observe that

$$b(d_h u(t), d_h u(t)) \geq c_3 \|d_h u(t)\|^2_{H_0^1(\Omega)}. \tag{15}$$

Therefore, (14) yields

$$c_3 \|d_h u(t)\|^2_{H_0^1(\Omega)} \leq \frac{1}{\lambda}\|d_h f(t)\|^2_{L^2(\Omega)} + \lambda \|d_h u(t)\|^2_{L^2(\Omega)} + \frac{1}{\lambda}\big\|d_h^{\tau} u(t)\big\|^2_{H_0^1(\Omega)} + \lambda\|d_h u(t)\|^2_{H_0^1(\Omega)}.$$

For sufficiently small $\lambda > 0$ there exists a constant $c_4 > 0$ such that

$$\|d_h u(t)\|^2_{H_0^1(\Omega)} \leq c_4\big(\|d_h f(t)\|^2_{L^2(\Omega)} + \big\|d_h^{\tau} u(t)\big\|^2_{H_0^1(\Omega)}\big). \tag{16}$$

Integrate (16) over t from 0 to $T-h$:

$$\int\limits_0^{T-h} \|d_h u(t)\|^2_{H_0^1(\Omega)}\,dt \leq c_4\Bigg(\int\limits_0^{T-h} \|d_h f(t)\|^2_{L^2(\Omega)}\,dt + \int\limits_0^{T-h} \big\|d_h^{\tau} u(t)\big\|^2_{H_0^1(\Omega)}\,dt\Bigg). \tag{17}$$

Observe that all smooth functions $v(t,x)$ satisfy

$$\int\limits_0^{T-h} \big\|d_h^{\tau} v(t)\big\|^2_{L^2(\Omega)}\,dt \leq \int\limits_0^{T} \|v(t)\|^2_{L^2(\Omega)}\,dt. \tag{18}$$

Taking this into account, we infer from (17) that

$$\int\limits_0^{T-h} \big\|d_h^{\tau} u(t)\big\|^2_{H_0^1(\Omega)}\,dt \leq c_4\Bigg(\int\limits_0^{T-h} \|d_h f(t)\|^2_{L^2(\Omega)}\,dt + \int\limits_0^{T} \|u(t)\|^2_{H_0^1(\Omega)}\,dt\Bigg). \tag{19}$$

Since $f_t(t) \in L^2(Q)$, we can look at (18) with $v = f_t$:

$$\int\limits_0^{T-h} \big\|d_h^{\tau} f_t(t)\big\|^2_{L^2(\Omega)}\,dt \leq \int\limits_0^{T} \|f_t(t)\|^2_{L^2(\Omega)}\,dt.$$

Then

$$\int_0^{T-h} \|d_h f(t)\|^2_{L^2(\Omega)}\,dt = \int_0^{T-h} \left\|\frac{f(t+h)-f(t)}{h}\right\|^2_{L^2(\Omega)} dt$$

$$= \int_0^{T-h} \left\|\frac{1}{h}\int_t^{t+h} f_\tau(\tau)\,d\tau\right\|^2_{L^2(\Omega)} dt = \int_0^{T-h} \left\|d_h^\tau f_t(t)\right\|^2_{L^2(\Omega)} dt \le \int_0^T \|f_t(t)\|^2_{L^2(\Omega)}\,dt.$$

Consequently, (19) yields

$$\int_0^{T-h} \|d_h u(t)\|^2_{H^1_0(\Omega)}\,dt \le c_4\Bigl(\int_0^T \|f_t(t)\|^2_{L^2(\Omega)}\,dt + \int_0^T \|u(t)\|^2_{H^1_0(\Omega)}\,dt\Bigr).$$

Suppose that h_0 is sufficiently small, but $h_0 \ge h$. Then

$$\int_0^{T-h_0} \|d_h u(t)\|^2_{H^1_0(\Omega)}\,dt \le c_4\Bigl(\int_0^T \|f_t(t)\|^2_{L^2(\Omega)}\,dt + \int_0^T \|u(t)\|^2_{H^1_0(\Omega)}\,dt\Bigr).$$

Passing to the limit as $h \to 0$, we obtain

$$\int_0^{T-h_0} \|u_t(t)\|^2_{H^1_0(\Omega)}\,dt \le c_4\Bigl(\int_0^T \|f_t(t)\|^2_{L^2(\Omega)}\,dt + \int_0^T \|u(t)\|^2_{H^1_0(\Omega)}\,dt\Bigr).$$

Since $h_0 \ge 0$ is arbitrary, it follows that

$$\|u_t(t)\|^2_{L^2(0,T;H^1(\Omega))} \le c_4(\|f_t(t)\|^2_{L^2(Q)} + \|u(t)\|^2_{L^2(0,T;H^1(\Omega))}). \tag{20}$$

Therefore, the derivative $u_t(t,x)$ exists; moreover, taking $v = u$ in (9), we obtain

$$\int_0^T\int_\Omega a_{ijkl}\varepsilon_{kl}(w(t))\varepsilon_{ij}(u(t))\,d\Omega dt = \int_0^T\int_\Omega f(t)u(t)\,d\Omega dt.$$

By (7),

$$\|u(t)\|^2_{L^2(0,T;H^{1,0}_\gamma(\Omega))} \le \frac{1}{\lambda}\|f(t)\|^2_{L^2(Q)} + \lambda\|u(t)\|^2_{L^2(0,T;H^{1,0}_\gamma(\Omega))}$$

and, for small $\lambda > 0$,

$$\|u(t)\|^2_{L^2(0,T;H^{1,0}_\gamma(\Omega))} \le c_5\|f(t)\|^2_{L^2(Q)}.$$

Then (20) yields

$$\|u_t(t)\|^2_{L^2(0,T;H^1(\Omega))} \le c(\|f_t(t)\|^2_{L^2(Q)} + \|f(t)\|^2_{L^2(Q)}).$$

The claim of the lemma follows.

To complete the proof of the theorem, we verify that problem (2)–(5) is equivalent to (6).

Proof of the theorem. According to Lemma 2, we can consider (6) for fixed $t \in (0,T)$:

$$b(w,v) = \int_\Omega fv\,d\Omega$$

with $w = w(t,x)$, $v = v(t,x)$, and $f = f(t,x)$. Rearrange the last equation as

$$\int\limits_{\Omega} \sigma_{ij}\varepsilon_{ij}(v)\,d\Omega = \int\limits_{\Omega} fv\,d\Omega. \tag{21}$$

Here, as above, we find σ_{ij} using (1), i.e., it involves integration from 0 to t. Insert $v \in C_0^\infty(\Omega)$, with $v = \rho$ on γ and $\rho \in R(\gamma)$, into (21) and integrate by parts. Then for this $t \in (0,T)$ the equations

$$-\frac{\partial \sigma_{ij}(t,x)}{\partial x_j} = f_i(t,x)$$

hold in the distribution sense.

Take $v \in H^{1,0}_\gamma(\Omega)$. Integrating (21) by parts yields

$$\int\limits_{\gamma} [\sigma_{ij}\nu_j]\,v_i\,d\gamma = 0, \quad v \in H^{1,0}_\gamma(\Omega),$$

that is, $v = \bar\rho$ on γ and $\bar\rho(x) \in R(\gamma)$ for fixed $t \in (0,T)$. We have

$$\int\limits_{\gamma} [\sigma_{ij}(t,x)\nu_j]\bar\rho_i(x)\,d\gamma = 0, \quad \bar\rho \in R(\gamma), \qquad \text{for almost all } t \in (0,T).$$

Conversely, multiplying (2) by $v \in H^{1,0}_\gamma(\Omega)$ and integrating by parts, we see that (5) implies (6).

Thus, requiring sufficient smoothness of solutions, we know that problems (2)–(5) and (6) are equivalent. Let us show how to give precise meaning to (5), even though the functions $\sigma_{ij}\nu_j$ are not defined on γ pointwise.

Extend γ to an intersection with Γ as indicated in the beginning of the article. Denote the extended curve by Σ; then $\gamma \subset \Sigma$.

Use the Green's formula [5, 18]

$$-\left\langle \frac{\partial \sigma_{ij}}{\partial x_j}, \bar u_i \right\rangle_D = \langle \sigma_{ij}, \varepsilon_{ij}(\bar u)\rangle_D - \langle \sigma_{ij} n_j, \bar u_i\rangle_{\frac12,\partial D}, \tag{22}$$

valid for all functions $\sigma_{ij}(x)$ with $\frac{\partial \sigma_{ij}(x)}{\partial x_j} \in L^2(D)$ for $i,j = 1,2$ and $\bar u \in H^1(D)$, where D is a domain with Lipschitz boundary, and $n = (n_1, n_2)$ is the unit vector of outer normal to ∂D. The brackets $\langle \cdot,\cdot\rangle_{\frac12,\partial D}$ stand for the duality between the space $H^{\frac12}(\partial D)$ and its dual $H^{-\frac12}(\partial D)$.

Observe that (22) holds in both cases $D = \Omega_i$ for $i = 1,2$ for the outer normal $n^i = \left(n_1^i, n_2^i\right)$ to $\partial\Omega_i$.

Introduce the space $H^{\frac12}_{00}(\Sigma)$ equipped with the norm

$$\|v\|^{00}_{\frac12,\Sigma} = \left(\|v\|^2_{\frac12,\Sigma} + \int\limits_{\Sigma} \frac{v^2}{r}\,d\Sigma \right)^{\frac12},$$

where $\|v\|^2_{\frac12,\Sigma}$ is the norm in $H^{\frac12}(\Sigma)$ and $r(x) = \mathrm{dist}(x, \partial\Sigma)$.

Assume that v is defined on Σ and denote by $\bar v$ the extension of v by 0 to $\partial\Omega_i$:

$$\bar v = \begin{cases} v & \text{on } \Sigma, \\ 0 & \text{on } \partial\Omega_i \setminus \Sigma. \end{cases}$$

Then $\bar v \in H^{\frac12}(\partial\Omega_i)$ if and only if $v \in H^{\frac12}_{00}(\Sigma)$.

Fix $t \in (0,T)$. Using (22), we infer from (21) that

$$-\left\langle \frac{\partial \sigma_{ij}}{\partial x_j}, v_i \right\rangle_{\Omega_1} + \left\langle \sigma_{ij} n_j^1, v_i \right\rangle_{\frac{1}{2},\partial\partial\Omega_1} - \left\langle \frac{\partial \sigma_{ij}}{\partial x_j}, v_i \right\rangle_{\Omega_2} + \left\langle \sigma_{ij} n_j^2, v_i \right\rangle_{\frac{1}{2},\partial\Omega_2} = \langle f, v\rangle_\Omega.$$

By (2), we obtain

$$\left\langle \sigma_{ij} n_j^1, v_i \right\rangle_{\frac{1}{2},\partial\Omega_1} + \left\langle \sigma_{ij} n_j^2, v_i \right\rangle_{\frac{1}{2},\partial\Omega_2} = 0.$$

Denoting by $H_{00}^{-\frac{1}{2}}(\Sigma)$ the dual space of $H_{00}^{\frac{1}{2}}(\Sigma)$, we can express this relation as

$$\langle [\sigma_{ij}\nu_j], v_i \rangle^{00}_{\frac{1}{2},\Sigma} = 0, \quad v \in H_\gamma^{1,0}(\Omega), \tag{23}$$

where the brackets $\langle \cdot,\cdot \rangle^{00}_{\frac{1}{2},\Sigma}$ stand for the duality between the spaces $H_{00}^{-\frac{1}{2}}(\Sigma)$ and $H_{00}^{\frac{1}{2}}(\Sigma)$.

Thus, (5) holds for almost all $t \in (0,T)$ in the sense of (23).

The proof of the theorem is complete.

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A MODIFIED BOUNDARY VALUE PROBLEM FOR STRONGLY DEGENERATE NONCLASSICAL DIFFERENTIAL EQUATIONS

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Abstract. We show that the strongly generalized nonclassical differential equations of higher order of the form

$$\sum_{k=0}^{2p} \alpha_k(t) D_t^{2p-k} u(x,t) - Au(x,t) = f(x,t)$$

become well posed on releasing part of the boundary of the domain from boundary value conditions.

Keywords: boundary value problem, nonclassical differential equations, generalized equations, elliptic operator

Assume that Ω is a bounded domain in $\mathbb{R}^n$ with smooth (for simplicity, infinitely differentiable) boundary Γ, $Q = \Omega \times (0,T)$ is a cylinder, $0 < T < +\infty$, $a^{ij}(x)$ $(i,j = 1,2,\dots,n)$, $a_0(x)$, $\alpha^k(t)$ $(k = 0,1,\dots,2p)$, and $f(x,t)$ are given functions of $x \in \overline{\Omega}$, $t \in [0,T]$, and $p > 1$ is an integer. Denote by D_t^l the derivative $\partial^l/\partial t^l$ and let A and L be the differential operators whose action on a function $v(x,t)$ is defined as

$$Av = \frac{\partial}{\partial x_i}(a^{ij}(x) v_{x_j}) + a_0(x) v$$

(here and in what follows, the repeated indices imply summation from 1 to n),

$$Lv = \sum_{k=0}^{2p} \alpha_k(t) D_t^{2p-k} v - Av;$$

below the operator A is assumed to be elliptic on $\overline{\Omega}$.

In Q we consider the equation

$$Lu = f(x,t) \tag{1}$$

where the unknown function is $u(x,t)$. For these equations in [1–3], the statements of boundary value problems are proposed and the existence and uniqueness theorems are proven for generalized and regular solutions. The main condition of these articles is the condition

$$(-1)^{p-1}[2\alpha_{2p-1}(t) + (1-2p)\alpha'_{2p}(t)] \geq \delta_0 > 0 \quad \text{for } t \in [0,T].$$

In this article we consider the situation when this condition fails and demonstrate that in some case we should change the statement of the problem as compared with that of [1–3], namely, we should reject some part of boundary conditions.

Let us proceed with the arguments and constructions for the case of $p = 2$.

Consider the equation

$$Lu = \alpha_0(t) D_t^4 u + \alpha_1(t) D_t^3 u + \alpha_2(t) D_t^2 u + \alpha_3(t) D_t u - Au = f(x,t). \tag{2}$$

Boundary Value Problem I. *Find a solution* $u(x,t)$ *to* (2) *in* Q *satisfying*

$$u(x,t)|_S = 0, \tag{3}$$

$$D_t^k u\big|_{t=0} = 0, \quad k = 0,1,2. \tag{4}$$

Boundary Value Problem II. *Find a solution* $u(x,t)$ *to* (2) *in* Q *satisfying* (3) *and* (4) *and such that*

$$D_t u|_{t=T} = 0. \tag{5}$$

Define the anisotropic Sobolev space

$$V = \big\{v(x,t) : v(x,t) \in L_2(0,T;W_2^2(\Omega)),\ D_t^k v \in L_2(Q),\ k = \overline{1,4}\big\}.$$

Introduce the notations:

$$h_1(t) = \frac{3}{2}\alpha_0'(t) - \alpha_1(t), \quad h_2(t) = -\frac{1}{2}\alpha_0'''(t) + \frac{1}{2}\alpha_1''(t) - \frac{1}{2}\alpha_2'(t) + \alpha_3(t),$$

$$h_3(t) = \frac{3}{2}\alpha_0''(t) - \alpha_1'(t) + \frac{1}{2}\alpha_2(t), \quad h_T = \alpha_0''(T) - \alpha_1'(T) + \alpha_2(T),$$

$$\tilde h_1(t) = h_1(t)(\lambda - t) - \frac{3}{2}\alpha_0(t), \quad \tilde h_2(t) = h_2(t)(\lambda - t) + h_3(t).$$

Lemma 1. *Assume that* $u(x,t)$ *is a solution to Problem* I *in* V *and*

$$f(x,t), f_t(x,t) \in L_2(Q), \tag{6}$$

$$a^{ij}(x) \in C^2(\overline{\Omega}),\ i,j = 1,\dots,n, \quad a_0(x) \in C(\overline{\Omega}), \tag{7}$$

$$a^{ij}(x) = a^{ji}(x),\ i,j = 1,\dots,n, \quad a^{ij}(x)\xi_i\xi_j \ge k_0|\xi|^2 \text{ for } x \in \overline{\Omega},\ k_0 > 0,\ \xi \in \mathbb{R}^n, \tag{8}$$

$$a_0(x) \le -\overline{a}_0 < 0, \tag{9}$$

$$\alpha_0(t), \alpha_3(t) \in C^3([0,T]), \quad \alpha_1(t), \alpha_2(t) \in C^2([0,T]), \tag{10}$$

$$\alpha_0(t) < 0 \quad \text{for } t \in [0,T), \tag{11}$$

$$\alpha_0(T) = \alpha_0'(T) = \alpha_1(T) = 0, \quad h_T \ge 0, \tag{12}$$

$$h_1(t) \ge 0, \quad h_2(t) \ge 0, \quad h_3(t) \ge 0, \quad t \in [0,T]. \tag{13}$$

Then

$$\int\limits_Q \tilde h_1(t)\big(D_t^2 u\big)^2\,dxdt + \int\limits_Q \tilde h_2(t)(D_t u)^2\,dxdt + \int\limits_Q u^2\,dxdt + \sum_{i=1}^n \int\limits_Q u_{x_i}^2\,dxdt \le M_1, \tag{14}$$

$$\sum_{i=1}^n \int\limits_Q \big[\tilde h_1(t)\big(D_t^2 u_{x_i}\big)^2 + \tilde h_2(D_t u_{x_i})^2\big]\,dxdt + \int\limits_Q (Au)^2\,dxdt \le M_2, \tag{15}$$

where the constants M_1 *and* M_2 *are determined by the functions* $f(x,t)$, $\alpha_0(t)$–$\alpha_3(t)$, $a^{ij}(x)$, $i,j = 1,2,\dots,n$, $a_0(x)$, *and the number* T.

Proof. To justify the first estimate, we multiply (2) by $(\lambda - t)D_t u$, $\lambda > T$, and integrate the result over Q. We find that

$$\int\limits_Q Lu(\lambda - t)D_t u\,dxdt = \int\limits_Q \alpha_0(t)D_t^4 u(\lambda - t)D_t u\,dxdt + \int\limits_Q \alpha_1(t)D_t^3 u(\lambda - t)D_t u\,dxdt$$

$$+ \int\limits_Q \alpha_2(t)D_t^2 u(\lambda - t)D_t u\,dxdt + \int\limits_Q \alpha_3(t)D_t u(\lambda - t)D_t u\,dxdt$$

$$+ \int\limits_Q Au(\lambda - t)D_t u\,dxdt = \int\limits_Q f(\lambda - t)D_t u\,dxdt.$$

Integrating and taking account of (3), (4), and (12), we arrive at the equality

$$\int\limits_Q \left[\left(\frac{3}{2}\alpha_0'(t)-\alpha_1(t)\right)(\lambda-t)-\frac{3}{2}\alpha_0(t)\right]\left(D_t^2u\right)^2 dxdt$$

$$+\int\limits_Q \left[\left(-\frac{1}{2}\alpha_0'''(t)+\frac{1}{2}\alpha_1''(t)-\frac{1}{2}\alpha_2'(t)+\alpha_3(t)\right)(\lambda-t)+\frac{3}{2}\alpha_0''(t)-\alpha_1'(t)\right.$$

$$\left.+\frac{1}{2}\alpha_2(t)\right](D_tu)^2\,dxdt-\frac{1}{2}\int\limits_Q a_0(x)u^2\,dxdt$$

$$+\frac{1}{2}\int\limits_Q a^{ij}(x)u_{x_i}u_{x_j}\,dxdt+\frac{\lambda-T}{2}\int\limits_\Omega[\alpha_0''(T)-\alpha_1'(T)+\alpha_2(T)](D_tu(x,T))^2\,dx$$

$$+(\lambda-T)\int\limits_\Omega a^{ij}(x)u_{x_j}(x,T)u_{x_i}(x,T)\,dx-\frac{\lambda-T}{2}\int\limits_\Omega a_0(x)(u(x,T))^2\,dx$$

$$=\int\limits_Q f(\lambda-t)D_tu\,dxdt.$$

Applying the Young inequality and the above notations, we infer

$$\int\limits_Q\left[h_1(t)(\lambda-t)-\frac{3}{2}\alpha_0(t)\right]\left(D_t^2u\right)^2dxdt+\int\limits_Q[h_2(t)(\lambda-t)+h_3(t)](D_tu)^2\,dxdt$$

$$-\frac{1}{2}\int\limits_Q a_0(x)u^2\,dxdt+\frac{1}{2}\int\limits_Q a^{ij}(x)u_{x_i}u_{x_j}\,dxdt+\frac{(\lambda-T)h_T}{2}\int\limits_\Omega(D_tu(x,T))^2\,dx$$

$$+(\lambda-T)\int\limits_\Omega a^{ij}(x)u_{x_j}(x,T)u_{x_i}(x,T)\,dx-\frac{\lambda-T}{2}\int\limits_\Omega a_0(x)u^2(x,T)\,dx$$

$$\leq\frac{\tilde\delta_1^2}{2}\int\limits_Q u^2\,dxdt+\frac{1}{2\tilde\delta_1^2}\int\limits_Q(\lambda-t)^2(D_tf)^2\,dxdt+\frac{\tilde\delta_2^2}{2}\int\limits_Q u^2\,dxdt+\frac{1}{2\tilde\delta_2^2}\int\limits_Q f^2\,dxdt$$

$$+\frac{(\lambda-T)\tilde\delta_3^2}{2}\int\limits_\Omega u^2(x,T)\,dx+\frac{1}{2\tilde\delta_3^2}\int\limits_\Omega f^2(x,T,)\,dx,$$

where $\tilde\delta_1$, $\tilde\delta_2$, and $\tilde\delta_3$ are arbitrary positive numbers. Choosing them small and employing (6)–(13), we arrive at (14).

To obtain the second estimate, we multiply (2) by $-(\lambda-t)Au_t$, $\lambda>T$, and integrate the result over Q. Thus,

$$-\int\limits_Q Lu(\lambda-t)Au_t\,dxdt=-\int\limits_Q\alpha_0(t)D_t^4u(\lambda-t)Au_t\,dxdt$$

$$-\int\limits_Q\alpha_1(t)D_t^3u(\lambda-t)Au_t\,dxdt-\int\limits_Q\alpha_2(t)D_t^2u(\lambda-t)Au_t\,dxdt$$

$$-\int\limits_Q\alpha_3(t)D_tu(\lambda-t)Au_t\,dxdt+\int\limits_Q Au(\lambda-t)Au_t\,dxdt=-\int\limits_Q f(\lambda-t)Au_t\,dxdt.$$

Integrating by parts, taking account of (3), (4), and (12), and applying the Young inequality and the above notations we arrive at the inequality

$$\int\limits_Q \left[h_1(t)(\lambda - t) - \frac{3}{2}\alpha_0(t)\right]\left(D_t^2 u_{x_i}\right)^2 dxdt + \int\limits_Q [h_2(t)(\lambda - t) + h_3(t)](D_t u_{x_i})^2 dxdt$$

$$+\frac{1}{2}\int\limits_Q (Au)^2 dxdt + \frac{(\lambda - T)}{2}\int\limits_{\partial Q}(Au(x,T))^2 ds$$

$$\leq \frac{\widetilde{\delta_4^2}}{2}\int\limits_Q (Au)^2 dxdt + \frac{1}{2\tilde{\delta}_4^2}\int\limits_Q (\lambda - t)^2 f_t^2 dxdt + \frac{\tilde{\delta}_5^2}{2}\int\limits_Q (Au)^2 dxdt + \frac{1}{2\tilde{\delta}_5^2}\int\limits_Q f^2 dxdt$$

$$+\frac{(\lambda - T)\widetilde{\delta_6^2}}{2}\int\limits_{\partial Q}(Au(x,T))^2 ds + \frac{1}{2\tilde{\delta}_6^2}\int\limits_{\partial Q} f^2(x,T)\, ds,$$

where $\tilde{\delta}_4$, $\tilde{\delta}_5$, and $\tilde{\delta}_6$ are arbitrary positive numbers. Choosing them small, we establish (15).

The lemma is proven.

Lemma 2. *Assume that $u(x,t)$ is a solution to Problem* I *in V, the conditions of Lemma* 1 *hold, and*

$$\begin{gathered}\alpha_3^4(t) \leq K_1\alpha_2^2(t), \quad \alpha_2^4(t) \leq K_2\alpha_3^2(t)\alpha_1^2(t), \quad (\alpha_2'(t))^2 \leq K_3\alpha_3^2(t),\\ \alpha_1^4(t) \leq K_4\alpha_2^2(t)\alpha_0^2(t), \quad (\alpha_1'(t))^2 \leq K_5\alpha_2^2(t), \quad K_i \geq 0,\ i = 1,\dots,5.\end{gathered} \tag{16}$$

Then

$$\int\limits_Q \alpha_0^2(t)\left(D_t^4 u\right)^2 dxdt \leq M_3, \tag{17}$$

where the constant M_3 is defined by the function $f(x,t)$ and the numbers K_1–K_5.

Proof. Multiply (2) by $\alpha_0(t)D_t^4 u$. In result we have

$$\int\limits_Q \alpha_0^2(t)\left[D_t^4 u\right]^2 dxdt = \int\limits_Q f(x,t)\alpha_0(t)D_t^4 u\, dxdt + \int\limits_Q Au\alpha_0(t)D_t^4 u\, dxdt$$

$$-\int\limits_Q \alpha_1(t)D_t^3 u\alpha_0(t)D_t^4 u\, dxdt - \int\limits_Q \alpha_2(t)D_t^2 u\alpha_0(t)D_t^4 u\, dxdt$$

$$-\int\limits_Q \alpha_3(t)D_t u\alpha_0(t)D_t^4 u\, dxdt.$$

Estimate the summands on the right-hand side with the use of the Young inequality.

We have

$$\int\limits_Q \alpha_0^2(t)\big[D_t^4u\big]^2\,dxdt \le \frac{\delta_0^2}{2}\int\limits_Q \alpha_0^2(t)\big[D_t^4u\big]^2\,dxdt + \frac{1}{2\delta_0^2}\int\limits_Q f^2(x,t)\,dxdt$$
$$+\frac{\delta_0^2}{2}\int\limits_Q \alpha_0^2(t)\big[D_t^4u\big]^2\,dxdt + \frac{1}{2\delta_0^2}\int\limits_Q [Au]^2\,dxdt + \frac{\delta_1^2}{2}\int\limits_Q \alpha_0^2(t)\big[D_t^4u\big]^2\,dxdt$$
$$+\frac{1}{2\delta_1^2}\int\limits_Q \alpha_1^2(t)\big[D_t^3u\big]^2\,dxdt + \frac{\delta_2^2}{2}\int\limits_Q \alpha_0^2(t)\big[D_t^4u\big]^2\,dxdt + \frac{1}{2\delta_2^2}\int\limits_Q \alpha_2^2(t)\big[D_t^2u\big]^2\,dxdt$$
$$+\frac{\delta_3^2}{2}\int\limits_Q \alpha_0^2(t)\big[D_t^4u\big]^2\,dxdt + \frac{1}{2\delta_3^2}\int\limits_Q \alpha_3^2(t)\big[D_tu\big]^2\,dxdt,$$

$$\int\limits_Q \alpha_0^2(t)\big[D_t^4u\big]^2\,dxdt \le \delta_0^2\int\limits_Q \alpha_0^2(t)\big[D_t^4u\big]^2\,dxdt + \frac{1}{2\delta_0^2}\int\limits_Q f^2(x,t)\,dxdt$$
$$+\frac{1}{2\delta_0^2}\int\limits_Q [Au]^2\,dxdt + \frac{\delta_1^2+\delta_2^2+\delta_3^2}{2}\int\limits_Q \alpha_0^2(t)\big[D_t^4u\big]^2\,dxdt$$
$$+\frac{1}{2\delta_1^2}I_1 + \frac{1}{2\delta_2^2}I_2 + \frac{1}{2\delta_3^2}I_3. \tag{18}$$

Estimate the integral I_3 as follows:

$$I_3 = \int\limits_Q \alpha_3^2(t)[D_tu]^2\,dxdt = -\int\limits_Q \alpha_3^2(t)uD_t^2u\,dxdt - 2\int\limits_Q \alpha_3(t)\alpha_3'(t)uD_tu\,dxdt$$
$$\le \frac{\delta_4^2}{2}\int\limits_Q \alpha_3^4(t)\big[D_t^2u\big]^2\,dxdt + \frac{1}{2\delta_4^2}\int\limits_Q u^2\,dxdt + \delta_5^2\int\limits_Q \alpha_3^2(t)[D_tu]^2\,dxdt$$
$$+\frac{1}{\delta_5^2}\int\limits_Q [\alpha_3'(t)]^2u^2\,dxdt,$$

$$\big(1-\delta_5^2\big)\int\limits_Q \alpha_3^2(t)[D_tu]^2\,dxdt \le \delta_4^2\int\limits_Q \alpha_3^4(t)\big[D_t^2u\big]^2\,dxdt$$
$$+\bigg(\frac{1}{\delta_4^2}\int\limits_Q u^2\,dxdt + \frac{2}{\delta_5^2}\int\limits_Q [\alpha_3'(t)]^2u^2\,dxdt\bigg).$$

Fix $\delta_5^2 = \frac{1}{2}$. From (16) it follows that

$$\int\limits_Q \alpha_3^2(t)[D_tu]^2\,dxdt \le \delta_4^2K_1\int\limits_Q \alpha_2^2(t)\big[D_t^2u\big]^2\,dxdt + C(\delta_4), \tag{19}$$

where the constant $C(\delta_4)$ is determined by the number δ_4 and the functions $\alpha_3(t)$ and $u(x,t)$.

Estimate I_2 as follows:

$$I_2 = \int\limits_Q \alpha_2^2(t)\big[D_t^2 u\big]^2\, dxdt = -\int\limits_Q \alpha_2^2(t) D_t^3 u D_t u\, dxdt - 2\int\limits_Q \alpha_2(t)\alpha_2'(t) D_t^2 u D_t u\, dxdt$$

$$= -\int\limits_Q \alpha_1^2(t) D_t^3 u \frac{\alpha_2^2(t)}{\alpha_1^2(t)} D_t u\, dxdt - 2\int\limits_Q \alpha_2(t) D_t^2 u \alpha_2'(t) D_t u\, dxdt$$

$$\le \frac{\delta_6^2}{2}\int\limits_Q \alpha_1^2(t)\big[D_t^3 u\big]^2\, dxdt + \frac{1}{2\delta_6^2}\int\limits_Q \frac{\alpha_2^4(t)}{\alpha_1^2(t)}[D_t u]^2\, dxdt$$

$$+\frac{\delta_7^2}{2}\int\limits_Q \alpha_2^2(t)\big[D_t^2 u\big]^2\, dxdt + \frac{1}{2\delta_7^2}\int\limits_Q [\alpha_2'(t)]^2 [D_t u]^2\, dxdt,$$

$$\big(2-\delta_7^2\big)\int\limits_Q \alpha_2^2(t)\big[D_t^2 u\big]^2\, dxdt \le \delta_6^2 \int\limits_Q \alpha_1^2(t)\big[D_t^3 u\big]^2\, dxdt$$

$$+\frac{1}{\delta_6^2(t)}\int\limits_Q \frac{\alpha_2^4(t)}{\alpha_1^2(t)}[D_t u]^2\, dxdt + \frac{1}{\delta_7^2}\int\limits_Q [\alpha_2'(t)]^2[D_t u]^2\, dxdt.$$

Let $\delta_7 = 1$. From (16) it follows that

$$\int\limits_Q \alpha_2^2(t)\big[D_t^2 u\big]^2\, dxdt \le \delta_6^2 \int\limits_Q \alpha_1^2(t)\big[D_t^3 u\big]^2\, dxdt$$

$$+\frac{K_2}{\delta_6^2}\int\limits_Q \alpha_3^2(t)[D_t u]^2\, dxdt + K_3\int\limits_Q \alpha_3^2(t)[D_t u]^2\, dxdt.$$

Employing (19), we find that

$$\int\limits_Q \alpha_2^2(t)\big[D_t^2 u\big]^2\, dxdt \le \delta_6^2 \int\limits_Q \alpha_1^2(t)\big[D_t^3 u\big]^2\, dxdt$$

$$+\frac{K_2}{\delta_6^2}\Bigg[\delta_4^2 K_1 \int\limits_Q \alpha_2^2(t)\big[D_t^2 u\big]^2\, dxdt + C(\delta_4)\Bigg]$$

$$+K_3\Bigg[\delta_4^2 K_1 \int\limits_Q \alpha_2^2(t)\big[D_t^2 u\big]^2\, dxdt + C(\delta_4)\Bigg]$$

or

$$\int\limits_Q \alpha_2^2(t)\big[D_t^2 u\big]^2\, dxdt \le \delta_6^2 \int\limits_Q \alpha_1^2(t)\big[D_t^3 u\big]^2\, dxdt$$

$$+\left(\frac{K_2}{\delta_6^2} + K_3\right) K_1\delta_4^2 \int\limits_Q \alpha_2^2(t)\big[D_t^2 u\big]^2\, dxdt + C_1(\delta_4,\delta_6). \tag{20}$$

Estimate I_1 as follows:

$$I_1 = \int\limits_Q \alpha_1^2(t)\big[D_t^3 u\big]^2\,dxdt = -\int\limits_Q \alpha_1^2(t) D_t^4 u D_t^2 u\,dxdt$$

$$-2\int\limits_Q \alpha_1(t)\alpha_1'(t) D_t^3 u D_t^2 u\,dxdt = -\int\limits_Q \alpha_0(t) D_t^4 u \frac{\alpha_1^2(t)}{\alpha_0(t)} D_t^2 u\,dxdt$$

$$-2\int\limits_Q \alpha_1(t) D_t^3 u \alpha_1'(t) D_t^2 u\,dxdt,$$

$$\int\limits_Q \alpha_1^2(t)\big[D_t^3 u\big]^2\,dxdt \le \frac{\delta_8^2}{2}\int\limits_Q \alpha_0^2(t)\big[D_t^4 u\big]^2\,dxdt + \frac{1}{2\delta_8^2}\int\limits_Q \frac{\alpha_1^4(t)}{\alpha_0^2(t)}\big[D_t^2 u\big]^2\,dxdt$$

$$+\delta_9^2\int\limits_Q \alpha_1^2(t)\big[D_t^3 u\big]^2\,dxdt + \frac{1}{\delta_9^2}\int\limits_Q [\alpha_1'(t)]^2\big[D_t^2 u\big]^2\,dxdt,$$

$$\big(1-\delta_9^2\big)\int\limits_Q \alpha_1^2(t)\big[D_t^3 u\big]^2\,dxdt \le \frac{\delta_8^2}{2}\int\limits_Q \alpha_0^2(t)\big[D_t^4 u\big]^2\,dxdt$$

$$+\frac{1}{2\delta_8^2}\int\limits_Q \frac{\alpha_1^4(t)}{\alpha_0^2(t)}\big[D_t^2 u\big]^2\,dxdt + \frac{1}{\delta_9^2}\int\limits_Q [\alpha_1'(t)]^2\big[D_t^2 u\big]^2\,dxdt.$$

Fix $\delta_9^2 = \frac{1}{2}$. From (16) it follows that

$$\int\limits_Q \alpha_1^2(t)\big[D_t^3 u\big]^2\,dxdt \le \delta_8^2\int\limits_Q \alpha_0^2(t)\big[D_t^4 u\big]^2\,dxdt$$

$$+\left(\frac{K_4}{\delta_8^2} + 4K_5\right)\int\limits_Q \alpha_2^2(t)\big[D_t^2 u\big]^2\,dxdt. \tag{21}$$

Estimating the first summand on the right-hand side of (20) with the use of (21), we arrive at the inequality

$$\int\limits_Q \alpha_2^2(t)\big[D_t^2 u\big]^2\,dxdt \le \delta_6^2\Bigg[\delta_8^2\int\limits_Q \alpha_0^2(t)\big[D_t^4 u\big]^2\,dxdt$$

$$+\left(\frac{K_4}{\delta_8^2} + 4K_5\right)\int\limits_Q \alpha_2^2(t)\big[D_t^2 u\big]^2\,dxdt\Bigg] + \left(\frac{K_2}{\delta_6^2} + K_3\right)K_1\delta_4^2\int\limits_Q \alpha_2^2(t)\big[D_t^2 u\big]^2\,dxdt$$

$$+C_1(\delta_4,\delta_6) = \delta_6^2\delta_8^2\int\limits_Q \alpha_0^2(t)\big[D_t^4 u\big]^2\,dxdt + \Bigg[\left(\frac{K_4}{\delta_8^2} + 4K_5\right)\delta_6^2$$

$$+\left(\frac{K_2}{\delta_6^2} + K_3\right)K_1\delta_4^2\Bigg]\int\limits_Q \alpha_2^2(t)\big[D_t^2 u\big]^2\,dxdt + C_1(\delta_4,\delta_6),$$

$$\left(1 - \frac{(K_4 + 4K_5\delta_8^2)\delta_6^4 + (K_2 + K_3\delta_6^2)K_1\delta_4^2\delta_8^2}{\delta_6^2\delta_8^2}\right)\int\limits_Q \alpha_2^2(t)\big[D_t^2 u\big]^2\,dxdt$$

$$\le \delta_6^2\delta_8^2\int\limits_Q \alpha_0^2(t)\big[D_t^4 u\big]^2\,dxdt + C_1(\delta_4,\delta_6),$$

$$\int\limits_Q \alpha_2^2(t)\big[D_t^2u\big]^2\,dxdt \le \left(\frac{\delta_6^4\delta_8^4}{\delta_6^2\delta_8^2-(K_4+4K_5\delta_8^2)\delta_6^4-(K_2+K_3\delta_6^2)K_1\delta_4^2\delta_8^2}\right)$$

$$\times\int\limits_Q \alpha_0^2(t)\big[D_t^4u\big]^2\,dxdt + C_2(\delta_4,\delta_6,\delta_8). \tag{22}$$

Estimate (21) with the help of (22) as follows:

$$\int\limits_Q \alpha_1^2(t)\big[D_t^3u\big]^2\,dxdt \le \delta_8^2\int\limits_Q \alpha_0^2(t)\big[D_t^4u\big]^2\,dxdt + \left(\frac{K_4}{\delta_8^2}+4K_5\right)$$

$$\times\Bigg[\left(\frac{\delta_6^4\delta_8^4}{\delta_6^2\delta_8^2-(K_4+4K_5\delta_8^2)\delta_6^4-(K_2+K_3\delta_6^2)K_1\delta_4^2\delta_8^2}\right)$$

$$\times\int\limits_Q \alpha_0^2(t)\big[D_t^4u\big]^2\,dxdt + C_2(\delta_4,\delta_6,\delta_8)\Bigg],\ \int\limits_Q \alpha_1^2(t)\big[D_t^3u\big]^2\,dxdt$$

$$\le\left(\delta_8^2+\frac{(K_4+4K_5\delta_8^2)\delta_6^4\delta_8^2}{\delta_6^2\delta_8^2-(K_4+4K_5\delta_8^2)\delta_6^4-(K_2+K_3\delta_6^2)K_1\delta_4^2\delta_8^2}\right)$$

$$\times\int\limits_Q \alpha_0^2(t)\big[D_t^4u\big]^2\,dxdt + C_3(\delta_4,\delta_6,\delta_8). \tag{23}$$

Next, we estimate (19) with the use of (22) as follows:

$$\int\limits_Q \alpha_3^2(t)[D_tu]^2\,dxdt \le \left(\frac{K_1\delta_4^2\delta_6^4\delta_8^4}{\delta_6^2\delta_8^2-(K_4+4K_5\delta_8^2)\delta_6^4-(K_2+K_3\delta_6^2)K_1\delta_4^2\delta_8^2}\right)$$

$$\times\int\limits_Q \alpha_0^2(t)\big[D_t^4u\big]^2\,dxdt + C_4(\delta_4,\delta_6,\delta_8). \tag{24}$$

Turning back to (18) and applying (22)–(24), we have

$$\int\limits_Q \alpha_0^2(t)\big[D_t^4u\big]^2\,dxdt \le \delta_0^2\int\limits_Q \alpha_0^2(t)\big[D_t^4u\big]^2\,dxdt$$

$$+\frac{1}{2\delta_0^2}\int\limits_Q f^2(x,t)\,dxdt + \frac{1}{2\delta_0^2}\int\limits_Q (Au)^2\,dxdt + \frac{\delta_1^2+\delta_2^2+\delta_3^2}{2}\int\limits_Q \alpha_0^2(t)\big[D_t^4u\big]^2\,dxdt$$

$$+\frac{1}{2\delta_1^2}\Bigg[\left(\delta_8^2+\frac{(K_4+4K_5\delta_8^2)\delta_6^4\delta_8^2}{\delta_6^2\delta_8^2-(K_4+4K_5\delta_8^2)\delta_6^4-(K_2+K_3\delta_6^2)K_1\delta_4^2\delta_8^2}\right)\int\limits_Q \alpha_0^2(t)\big[D_t^4u\big]^2\,dxdt$$

$$+C_3(\delta_4,\delta_6,\delta_8)\Bigg] + \frac{1}{2\delta_2^2}\Bigg[\left(\frac{\delta_6^4\delta_8^4}{\delta_6^2\delta_8^2-(K_4+4K_5\delta_8^2)\delta_6^4-(K_2+K_3\delta_6^2)K_1\delta_4^2\delta_8^2}\right)$$

$$\times\int\limits_Q \alpha_0^2(t)\big[D_t^4u\big]^2\,dxdt + C_2(\delta_4,\delta_6,\delta_8)\Bigg]$$

$$+\frac{1}{2\delta_3^2}\Bigg[\left(\frac{K_1\delta_4^2\delta_6^4\delta_8^4}{\delta_6^2\delta_8^2-(K_4+4K_5\delta_8^2)\delta_6^4-(K_2+K_3\delta_6^2)K_1\delta_4^2\delta_8^2}\right)$$

$$\times\int\limits_Q \alpha_0^2(t)\big[D_t^4u\big]^2\,dxdt + C_4(\delta_4,\delta_6,\delta_8)\Bigg].$$

Assume that $\delta_1 = \delta_2 = \delta_3$, $\delta_6 = \delta_8^2$, and $\delta_4 = \delta_8^4$. In this case we find that

$$\int\limits_Q \alpha_0^2(t)\big[D_t^4 u\big]^2\,dxdt \le \delta_0^2 \int\limits_Q \alpha_0^2(t)\big[D_t^4 u\big]^2\,dxdt + \frac{3\delta_1^2}{2}\int\limits_Q \alpha_0^2(t)\big[D_t^4 u\big]^2\,dxdt$$

$$+\frac{1}{2\delta_1^2}\left[\delta_8^2 + \frac{(K_4+4K_5\delta_8^2)\delta_8^{10}+\delta_8^{12}+K_1\delta_8^{20}}{\delta_8^6-(K_4+4K_5\delta_8^2)\delta_8^8-(K_2+K_3\delta_8^4)K_1\delta_8^{10}}\right]\int\limits_Q \alpha_0^2(t)\big[D_t^4 u\big]^2\,dxdt$$

$$+\frac{1}{2\delta_1^2}\left[C_2(\delta_8)+C_3(\delta_8)+C_4(\delta_8)\right]+\frac{1}{2\delta_0^2}\int\limits_Q f^2(x,t)\,dxdt+\frac{1}{2\delta_0^2}\int\limits_Q (Au)^2\,dxdt. \qquad (25)$$

Choose δ_8 so small that

$$1-(K_1K_2+K_4)\delta_8^2 > \frac{1}{2}. \qquad (26)$$

We have

$$\frac{1}{\delta_8^6(1-(K_4+4K_5\delta_8^2)\delta_8^2-(K_2+K_3\delta_8^4)K_1\delta_8^4)} < \frac{2}{\delta_8^6}.$$

Thus,

$$\delta_8^2 + \frac{(K_4+4K_5\delta_8^2)\delta_8^{10}+\delta_8^{12}+K_1\delta_8^{20}}{\delta_8^6-(K_4+4K_5\delta_8^2)\delta_8^8-(K_2+K_3\delta_8^4)K_1\delta_8^{10}} < \delta_8^2 + \frac{2(\delta_8^{10}+\delta_8^{12}+\delta_8^{20})}{\delta_8^6} < 7\delta_8^2.$$

Fix $\delta_0^2 = \frac{1}{6}$ and $\delta_1^2 = \frac{1}{9}$. From (25) and (26) we obtain that

$$\delta_8 = \min\left\{\frac{1}{3\sqrt{21}};\frac{1}{\sqrt{2(K_1K_2+K_4)}}\right\}.$$

Denote $C_0(\delta_8) = \left(\frac{1}{3}+\frac{63}{2}\delta_8^2\right)$. In this case we infer

$$\int\limits_Q \alpha_0^2(t)\big[D_t^4 u\big]^2\,dxdt \le C_0(\delta_8)\int\limits_Q \alpha_0^2(t)\big[D_t^4 u\big]^2\,dxdt + \frac{9}{2}\left[C_2(\delta_8)+C_3(\delta_8)+C_4(\delta_8)\right]$$

$$+3\int\limits_Q f^2(x,t)\,dxdt + 3\int\limits_Q (Au)^2\,dxdt.$$

Since $C_0(\delta_8) < 1$, in view of (15) we justify the required estimate (17).

The lemma is proven.

Corollary. *Under the conditions of Lemma* 2, *a solution $u(x,t)$ to Problem* I *in V satisfies the estimates*

$$\int\limits_Q \left[\alpha_1^2(t)\big(D_t^3 u\big)^2+\alpha_2^2(t)\big(D_t^2 u\big)^2+\alpha_3^2(t)(D_t u)^2\right]dxdt \le M_4, \qquad (27)$$

$$\sum_{i,j=1}^n \int\limits_Q u_{x_ix_j}^2\,dxdt \le M_5, \qquad (28)$$

where the constants M_4 and M_5 are defined by the function $f(x,t)$ and the numbers K_1–K_5.

Define the Sobolev space

$$V_0 = \left\{v(x,t) : v(x,t)\in L_2\big(0,T;W_2^2(\Omega)\big),\ |\alpha_k(t)|D_t^{4-k}v\in L_2(Q),\ k=\overline{1,3}\right\}.$$

Theorem 1. *Let the conditions of Lemma* 2 *hold. Then there exists a solution* $u(x,t)$ *to Problem* I *in* V_0.

PROOF. Assume that ε is a positive number and $L_\varepsilon u$ is an operator of the form

$$L_\varepsilon u = (\alpha_0(t)-\varepsilon)D_t^4 u + \alpha_1(t)D_t^3 u + \alpha_2(t)D_t^2 u + \alpha_3(t)D_t u - Au - \varepsilon A u_t.$$

Consider the following family of problems: *Find a solution* $u(x,t)$ *to*

$$L_\varepsilon u = f(x,t)$$

satisfying (3) *and* (4) *and such that*

$$D_t u|_{t=T} = 0, \quad x \in \Omega.$$

It follows from [1–3] that this problem has a solution $u^\varepsilon(x,t)$ from V. The functions $\{u^\varepsilon(x,t)\}$ satisfy the estimates

$$\int\limits_Q \left(\tilde{h}_1(t) + \frac{3}{2}\varepsilon\right)\left(D_t^2 u\right)^2 dxdt + \int\limits_Q \tilde{h}_2(t)(D_t u)^2\, dxdt + \int\limits_Q u^2\, dxdt$$
$$+(1+\varepsilon)\sum_{i=1}^n \int\limits_Q u_{x_i}^2\, dxdt \le M_6, \qquad (14_\varepsilon)$$

$$\sum_{i=1}^n \int\limits_Q \left[\left(\tilde{h}_1(t)+\frac{3}{2}\varepsilon\right)\left(D_t^2 u_{x_i}\right)^2 + \tilde{h}_2(D_t u_{x_i})^2\right] dxdt + \int\limits_Q (Au)^2\, dxdt$$
$$+\varepsilon \int\limits_Q (\lambda - t)(Au_t)^2\, dxdt \le M_7, \qquad (15_\varepsilon)$$

$$\int\limits_Q \alpha_0^2(t)\left(D_t^4 u\right)^2 dxdt + \varepsilon \int\limits_Q \alpha_0(t)\left(D_t^4 u\right)^2 dxdt + \varepsilon^2 \int\limits_Q (Au_t)^2\, dxdt \le M_8, \qquad (16_\varepsilon)$$

$$\int\limits_Q \left[\alpha_1^2(t)\left(D_t^3 u\right)^2 + \alpha_2^2(t)\left(D_t^2 u\right)^2 + \alpha_3^2(t)(D_t u)^2\right] dxdt \le M_9, \qquad (27_\varepsilon)$$

$$\sum_{i,j=1}^n \int\limits_Q u_{x_i x_j}^2\, dxdt \le M_{10}. \qquad (28_\varepsilon)$$

We can validate these estimates with arguments of Lemmas 1 and 2 (also see the corollary to Lemma 2). Since every Hilbert space is reflexive [4], the estimates obtained imply that there exists a sequence of positive integers $\{m_l\}$ and a function $u(x,t)$ such that $u(x,t) \in V_0$ and as $l \to \infty$ we have

$$u^{\varepsilon_{m_l}}(x,t) \to u(x,t), \quad u_{x_i}^{\varepsilon_{m_l}}(x,t) \to u_{x_i}(x,t), \quad u_{x_i x_j}^{\varepsilon_{m_l}}(x,t) \to u_{x_i x_j}(x,t),\ i,j = \overline{1,n},$$
$$|\alpha_k(t)| D_t^{4-k} u_t^{\varepsilon_{m_l}}(x,t) \to |\alpha_k(t)| D_t^{4-k} u_t(x,t), \quad k = \overline{1,3},$$
$$\varepsilon_{m_l} D_t^4 u^{m_l}(x,t) \to 0, \quad \varepsilon_{m_l} A u_t^{m_l}(x,t) \to 0$$

converges weakly as $m_l \to \infty$ in $L_2(Q)$. These convergences imply that $u(x,t)$ is a solution to (2)–(4) in V_0.

The theorem is proven.

Assume that $\alpha_k(t)$ $(k=\overline{0,3})$ are the functions $(T-t)^{p_k}$ with $p_k \geq 0$ integers. As is easily seen, we can find numbers p_k such that the conditions of Lemma 2 hold but the condition of [1–3] fails.

Introduce the notations:

$$\varphi_1(t) = (\alpha_0(t)\alpha_1(t))' + 2\alpha_0(t)\alpha_2(t), \quad \varphi_2(t) = (\alpha_0(t)\alpha_2(t))'' + 3(\alpha_0(t)\alpha_3(t))',$$

$$\varphi_3(t) = (\alpha_0(t)\alpha_3(t))'''.$$

Lemma 3. *Assume that $u(x,t)$ is a solution to Problem* I *in V, the conditions of Lemma* 1 *hold, and*

$$\varphi_1(t) \leq 0, \quad \varphi_2(t) \geq 0, \quad \varphi_3(t) \leq 0, \quad \alpha_1(0) \geq 0, \quad \alpha_0''(T)\alpha_3(T) \geq 0. \tag{29}$$

Then

$$\int\limits_Q \alpha_0^2(t)\big(D_t^4 u\big)^2\,dxdt + \int\limits_Q |\varphi_1(t)|\big(D_t^3 u\big)^2\,dxdt +$$
$$+ \int\limits_Q \varphi_2(t)\big(D_t^2 u\big)^2\,dxdt + \int\limits_Q |\varphi_3(t)|(D_t u)^2\,dxdt \leq M_{11}, \tag{30}$$

where the constant M_{11} is defined by the functions $f(x,t)$ and $\alpha_0(t)$–$\alpha_3(t)$.

Proof. Multiplying (2) by $\alpha_0(t)D_t^4 u$, we obtain the equality

$$\int\limits_Q \alpha_0^2(t)\big(D_t^4 u\big)^2\,dxdt + \int\limits_Q \alpha_0(t)\alpha_1(t)D_t^3 u D_t^4 u\,dxdt + \int\limits_Q \alpha_0(t)\alpha_2(t)D_t^2 u D_t^4 u\,dxdt$$
$$+ \int\limits_Q \alpha_0(t)\alpha_3(t)D_t u D_t^4 u\,dxdt - \int\limits_Q \alpha_0(t)Au D_t^4 u\,dxdt = \int\limits_Q f(x,t)\alpha_0(t)D_t^4 u\,dxdt.$$

Integrating by parts and involving the Young inequality and the conditions of Lemma 1, we arrive at the inequality

$$\int\limits_Q \alpha_0^2(t)\big(D_t^4 u\big)^2\,dxdt - \frac{1}{2}\int\limits_Q [(\alpha_0(t)\alpha_1(t))' + 2\alpha_0(t)\alpha_2(t)]\big(D_t^3 u\big)^2\,dxdt$$
$$+\frac{1}{2}\int\limits_Q [(\alpha_0(t)\alpha_2(t))'' + 3(\alpha_0(t)\alpha_3(t))']\big(D_t^2 u\big)^2\,dxdt - \frac{1}{2}\int\limits_Q (\alpha_0(t)\alpha_3(t))'''(D_t u)^2\,dxdt$$
$$-\frac{1}{2}\int\limits_\Omega \alpha_0(0)\alpha_1(0)\big(D_t^3 u(x,0)\big)^2\,dx + \frac{1}{2}\int\limits_\Omega \alpha_0''(T)\alpha_3(T)(D_t u(x,T))^2\,dx$$
$$\leq \frac{\delta_0^2}{2}\int\limits_Q \alpha_0^2(t)\big(D_t^4 u\big)^2\,dxdt + \frac{1}{2\delta_0^2}\int\limits_Q (Au)^2\,dxdt$$
$$+\frac{\delta_1^2}{2}\int\limits_Q \alpha_0^2(t)\big(D_t^4 u\big)^2\,dxdt + \frac{1}{2\delta_1^2}\int\limits_Q f^2(x,t)\,dxdt,$$

where δ_0 and δ_1 are arbitrary positive numbers. Choosing these constants sufficiently small and taking (15) into account, we establish (30). The lemma is proven.

Theorem 2. *Let the conditions of Lemma* 3 *hold. Then there exists a solution* $u(x,t)$ *to Problem* I *in* V_0.

PROOF. Use the method of ε-regularization as in the proof of Theorem 1.

We turn to solvability of Boundary Value Problem II. Assume now that

$$\alpha_0(t) \le 0 \quad \text{for } t \in (0,T), \quad \alpha_0(0) < 0, \quad \alpha_0(T) < 0. \tag{31}$$

Lemma 4. *Assume that* $u(x,t)$ *is a solution to Problem* II *in* V *satisfying* (6)–(10), (13), *and* (31). *Then* (14) *and* (15) *are fulfilled.*

PROOF. Argue by analogy with the proof of Lemma 1.

Lemma 5. *Assume that* $u(x,t)$ *is a solution to Problem* II *in* V *satisfying the conditions of Lemma* 4 *as well as the conditions*

$$|\varphi_1(t)| \le C_1\alpha_1^2(t), \quad \varphi_2(t) \le C_2\alpha_2^2(t), \quad |\varphi_3(t)| \le C_3\alpha_3^2(t), \quad C_i \ge 0,\ i = \overline{1,3},$$
$$\alpha_0(T)\alpha_1(T)\xi^2 - 2\alpha_0(T)\alpha_2(T)\xi\eta - [(\alpha_0(T)\alpha_2(T))' + \alpha_0(T)\alpha_3(T)]\eta^2 \ge 0, \tag{32}$$
$$\alpha_0(0) \ge 0, \quad (\xi,\eta) \in \mathbb{R}^2.$$

Then the estimate

$$\int_Q \alpha_0^2(t)\big(D_t^4u\big)^2\,dxdt + \int_Q \alpha_1^2(t)\big(D_t^3u\big)^2\,dxdt$$
$$+ \int_Q \alpha_2^2(t)\big(D_t^2u\big)^2\,dxdt + \int_Q \alpha_3^2(t)(D_tu)^2\,dxdt \le M_{12} \tag{33}$$

is fulfilled, where M_{12} *is defined by the functions* $f(x,t)$, $\alpha_0(t)$–$\alpha_3(t)$ *and the numbers* C_1–C_3.

PROOF. Multiply (2) by $\alpha_0(t)D_t^4u$. Integrating by parts and involving the Young inequality, we arrive at the inequality

$$\int_Q \alpha_0^2(t)\big(D_t^4u\big)^2\,dxdt - \frac{1}{2}\int_Q [(\alpha_0(t)\alpha_1(t))' + 2\alpha_0(t)\alpha_2(t)]\big(D_t^3u\big)^2\,dxdt$$
$$+\frac{1}{2}\int_Q [(\alpha_0(t)\alpha_2(t))'' + 3(\alpha_0(t)\alpha_3(t))']\big(D_t^2u\big)^2\,dxdt - \frac{1}{2}\int_Q (\alpha_0(t)\alpha_3(t))'''(D_tu)^2\,dxdt$$
$$+\frac{1}{2}\int_\Omega \alpha_0(T)\alpha_1(T)\big(D_t^3u(x,T)\big)^2\,dx - \frac{1}{2}\int_\Omega \alpha_0(0)\alpha_1(0)\big(D_t^3u(x,0)\big)^2\,dx$$
$$-\int_\Omega \alpha_0(T)\alpha_2(T)D_t^2u(x,T)D_t^3u(x,T)\,dx - \frac{1}{2}\int_\Omega (\alpha_0(T)\alpha_2(T))'\big(D_t^2u(x,T)\big)^2\,dx$$
$$-\frac{1}{2}\int_\Omega \alpha_0(T)\alpha_3(T)\big(D_t^2u(x,T)\big)^2\,dx \le \frac{\delta_0^2}{2}\int_Q \alpha_0^2(t)\big(D_t^4u\big)^2\,dxdt$$
$$+\frac{1}{2\delta_0^2}\int_Q (Au)^2\,dxdt + \frac{\delta_1^2}{2}\int_Q \alpha_0^2(t)\big(D_t^4u\big)^2\,dxdt + \frac{1}{2\delta_1^2}\int_Q f^2(x,t)\,dxdt,$$

where δ_0 and δ_1 are arbitrary positive numbers. Choosing these numbers small and taking (15) and (32) into account, we obtain (33).

The lemma is proven.

Theorem 3. *Let the conditions of Lemma* 5 *hold. Then there exists a solution* $u(x, t)$ *to Problem* II *in* V_0*.*

Proof. Argue by analogy to the proof of Theorem 1.

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ON SOME NONLOCAL PROBLEMS FOR THIRD ORDER EQUATIONS WITH MULTIPLE CHARACTERISTICS

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Abstract. We consider two boundary value problems for an equation of the third order with multiple characteristics and a nonlocal condition in time. In order to prove uniqueness, we use the method of energy integrals. By the method of potentials, the Green's function is constructed and employed to prove the unique solvability of the problems in question. The influence is studied of the boundary conditions on smoothness of solutions.

Keywords: third order equations with multiple characteristics, Green's function, the method of energy integrals, boundary value problem, nonlocal condition, integral equation

1. Introduction

In the article [1] a method of constructing fundamental solutions for equations with multiple characteristics was elaborated, and the fundamental solutions for the equations

$$Lu \equiv \frac{\partial^3 u}{\partial x^3} - \frac{\partial u}{\partial t} = 0, \tag{1}$$

$$Lu \equiv \frac{\partial^3 u}{\partial x^3} - \frac{\partial^2 u}{\partial t^2} = 0 \tag{2}$$

were constructed. At the beginning of the 1960s, Cattabriga following the article by Del Vecchio [1] constructed the potential theory for the above equations [2, 3]. Further, a series of boundary value problems for (1) and (2) with local and nonlocal boundary conditions was considered in [2–13].

In the present article we consider the problems:

I. Find a regular solution $u(x,t) \in K_u$ to (1) in the domain $\Omega = \{(x,t) : 0 < x < 1,\ 0 < t \le T\}$ satisfying the conditions

$$u(x,0) = \mu u(x,T), \quad \mu = \text{const},\ 0 \le x \le 1, \tag{3}$$

$$u(0,t) = \varphi_1(t), \quad u_x(0,t) = \varphi_2(t), \quad u_x(1,t) = \psi(t), \quad 0 \le t \le T. \tag{4}$$

Here $K_u = \big\{u(x,t) : u(x,t) \in C^{3,1}_{x,t}(\Omega) \cap C^{2,0}_{x,t}(\overline{\Omega}),\ u_{xt} \in C(\Omega)\big\}$.

II. Find a regular solution $u(x,t) \in K_u$ to (1) in the domain $\Omega = \{(x,t) : 0 < x < 1,\ 0 < t \le T\}$ satisfying the conditions

$$u(x,0) = \mu u(x,T), \quad \mu = \text{const},\ 0 \le x \le 1, \tag{5}$$

$$u_{xx}(0,t) = \varphi_1(t), \quad u_x(0,t) = \varphi_2(t), \quad u_x(1,t) = \psi(t), \quad 0 \le t \le T. \tag{6}$$

Here $\widetilde{K}_u = \big\{u(x,t) : u(x,t) \in C^{4,1}_{x,t}(\Omega) \cap C^{3,1}_{x,t}(\overline{\Omega}),\ u_{xt} \in C(\Omega)\big\}$.

Note that the nonlocal problem for the equation

$$Lu \equiv \frac{\partial^3 u}{\partial x^3} + \frac{\partial u}{\partial t} + \gamma u = 0$$

is studied with the other boundary conditions in [14], where to prove solvability of the problem the authors used the method of parabolic regularization and the method of continuation in a parameter.

At this article we use the methods of potentials.

The fundamental solutions to (1) are of the form (see [3])

$$U(x-\xi; t-\tau) = (t-\tau)^{-\frac{1}{3}} f\left(\frac{x-\xi}{(t-\tau)^{\frac{1}{3}}}\right), \quad x \neq \xi,\ t > \tau, \tag{7}$$

$$V(x-\xi; t-\tau) = (t-\tau)^{-\frac{1}{3}} \varphi\left(\frac{x-\xi}{(t-\tau)^{\frac{1}{3}}}\right), \quad x > \xi,\ t > \tau. \tag{8}$$

Here

$$f(z) = \int_0^\infty \cos(\lambda^3 - \lambda z)\, d\lambda, \quad -\infty < z < \infty,$$

$$\varphi(z) = \int_0^\infty (\exp(-\lambda^3 - \lambda z) + \sin(\lambda^3 - \lambda z))\, d\lambda, \quad z > 0,$$

$$z = (x-\xi)(t-\tau)^{-\frac{1}{3}}.$$

The functions $U(x-\xi; t-\tau)$ and $V(x-\xi; t-\tau)$, $f(z)$, and $\varphi(z)$ meet the relations (see [3])

$$f''(z) + \frac{1}{3} z f(z) = 0, \quad \varphi''(z) + \frac{1}{3} z \varphi(z) = 0, \tag{9}$$

$$\int_{-\infty}^{\infty} f(z) = \pi, \quad \int_{-\infty}^{0} f(z) = \frac{\pi}{3}, \quad \int_{0}^{\infty} f(z) = \frac{2\pi}{3}, \quad \int_{0}^{\infty} \varphi(z) = 0, \tag{10}$$

$$\lim_{(x,t)\to(a-0,t)} \int_\tau^t U_{\xi\xi}(x-a; t-\tau)\alpha(\xi,\tau)\, d\tau = \frac{\pi}{3}\alpha(t), \tag{11}$$

$$\lim_{(x,t)\to(a+0,t)} \int_\tau^t U_{\xi\xi}(x-a; t-\tau)\alpha(\xi,\tau)\, d\tau = -\frac{2\pi}{3}\alpha(t), \tag{12}$$

$$\lim_{(x,t)\to(a+0,t)} \int_\tau^t V_{\xi\xi}(x-a; t-\tau)\alpha(\xi,\tau)\, d\tau = 0, \tag{13}$$

$$f^n(z) \sim c_n^+ z^{\frac{2n-1}{4}} \sin\left(\frac{2}{3} z^{\frac{3}{2}}\right) \quad \text{as } z \to \infty, \tag{14}$$

$$\varphi^n(z) \sim c_n^+ z^{\frac{2n-1}{4}} \sin\left(\frac{2}{3} z^{\frac{3}{2}}\right) \quad \text{as } z \to \infty, \tag{15}$$

$$f^n(z) \sim c_n^- |z|^{\frac{2n-1}{4}} \exp\left(-\frac{2}{3}|z|^{\frac{3}{2}}\right) \quad \text{as } z \to -\infty. \tag{16}$$

2. The Main Results

Theorem 1. *Let* $0 \le \mu^2 \le \exp\{-T\}$. *Then Problems* I *and* II *have at most one solution.*

Proof. First, we prove that Problem I has a unique solution. Let Problem I have two solutions $u_1(x,t)$ and $u_2(x,t)$. Assigning $v(x,t) = u_1(x,t) - u_2(x,t)$, we obtain the following problem for $v(x,t)$:

$$Lv \equiv \frac{\partial^3 v}{\partial x^3} - \frac{\partial v}{\partial t} = 0, \quad (x,t) \in \Omega, \tag{17}$$

$$v(x,0) = \mu v(x,T), \quad 0 \le x \le 1, \tag{18}$$

$$v(0,t) = 0, \quad v_x(0,t) = 0, \quad v_x(1,t) = 0, \quad 0 \le t \le T. \tag{19}$$

Consider the identity

$$\int_0^1 \int_0^T Lv \; v_{xt} \exp\{-t\} \, dxdt = 0. \tag{20}$$

Integrating by parts in (20) and accounting for the homogeneous boundary conditions (18), (19), we see that

$$-\frac{1}{2} \int_0^1 \int_0^T v_{xx}^2(x,t) \exp\{-t\} \, dxdt - \frac{1}{2} \int_0^T v_t^2(1,t) \exp\{-t\} \, dt$$
$$-\frac{1}{2} \int_0^1 v_{xx}^2(x,T) \{\exp\{-T\} - \mu^2\} \, dx = 0. \tag{21}$$

Hence, $v_{xx}(x,t) = 0$ in Ω and $v_t(1,t) = 0$ for $t \in [0,T]$.

Let $\mu^2 < \exp\{-T\}$. In this case $v_{xx}(x,T) = 0$ and so $v_x(x,T) = \text{const}$, $x \in [0,1]$. Since

$$v_x(0,t) = v_x(1,t) = 0, \quad t \in [0,T],$$

we have

$$v_x(0,0) = v_x(0,T) = 0.$$

Therefore, $v_x(x,T) = \text{const} = 0$ for $x \in [0,1]$.

Next,

$$v_x(x,T) = 0 \Rightarrow v(x,T) = \text{const} \Rightarrow v(x,0) = \text{const}, \quad x \in [0,1].$$

Since $v(0,t) = 0 \Rightarrow v(0,0) = 0$, $v(x,0) = \text{const} = 0$ for $x \in [0,1]$. Moreover, we have $v_t(1,t) = 0$ for $t \in [0,T] \Rightarrow v(1,t) = \text{const}$ for $t \in [0,T]$ and $v(0,0) = 0$; thus $v(1,t) = 0$, $t \in [0,T]$.

We arrive at the well-known first boundary value problem for $v(x,t)$; i.e.,

$$Lv \equiv \frac{\partial^3 v}{\partial x^3} - \frac{\partial v}{\partial t} = 0, \quad (x,t) \in \Omega,$$

$$v(x,0) = 0, \; x \in [0,1], \quad v(0,t) = 0, \quad v_x(0,t) = 0, \quad v(1,t) = 0, \; t \in [0,T].$$

In view of [3] this problem is uniquely solvable.

Consider the case of $\mu^2 = \exp\{-T\}$. From (20) it follows that $v_{xx}(x,t) = 0$ and so $v_x(x,t) = \delta_1(t)$ for $t \in [0,T]$. Since $v_x(0,t) = v_x(1,t) = 0$ for $t \in [0,T]$, $\delta_1(t) = 0$ for $t \in [0,T]$.

The equality $v_x(x,t) = 0$ implies that $v(x,t) = \delta_2(t)$ for $t \in [0,T]$. Since $v(0,t) = 0$ for $t \in [0,T]$, $\delta_2(t) = 0$ for $t \in [0,T]$.

Since $v(x,t) = 0$ is continuous, we find that $v(x,t) = 0$ in $\overline{\Omega}$.

Consider Problem II. Let Problem II have two solutions $u_1(x,t)$ and $u_2(x,t)$. Putting $v(x,t) = u_1(x,t) - u_2(x,t)$, we arrive at the homogeneous problem of the form similar to (2) for $v(x,t)$. Put $\omega(x,t) = v_x(x,t)$. Differentiating (1), we infer

$$L\omega \equiv \frac{\partial^3 \omega}{\partial x^3} - \frac{\partial \omega}{\partial t} = 0, \quad (x,t) \in \Omega,$$

$$\omega(x,0) = \mu\omega(x,T), \quad 0 \le x \le 1,$$

$$\omega(0,t) = 0, \quad \omega_x(0,t) = 0, \quad \omega(1,t) = 0, \quad 0 \le t \le T.$$

Integrating the identity

$$\int_0^1 \int_0^T L\omega \; \omega \exp\{-t\} \, dxdt = 0,$$

we find that $\omega(x,t) = 0$ for $(x,t) \in \overline{\Omega}$, $\omega_x(1,t) = 0$ for all $t \in [0,T]$, and $\omega(x,T) = 0$ for $x \in [0,1]$.

Integrating the identity

$$\int_0^1 \int_0^T L\omega \; \omega \exp\{-x\} \, dxdt = 0,$$

we have that $\omega_x(x,t) = 0$ for all $(x,t) \in \overline{\Omega}$. In this case $v_x(x,t) = 0$ and $v_{xx}(x,t) = 0$ for all $(x,t) \in \overline{\Omega}$.

Consider the identity

$$\int_0^1 \int_0^T Lv \; v \exp\{-t\} \, dxdt = 0.$$

Integrating it, we establish that $v(x,t) = 0$ for all $(x,t) \in \overline{\Omega}$.

Now we prove the existence theorem for solutions to Problem I.

Theorem 2. *Assume that* $0 \le \mu^2 \le \exp\{-T\}$, $\psi(t) \in C^1([0,T])$, $\varphi_2(t) \in C^1([0,T])$, *and* $\varphi_1(t) \in C^2([0,T])$. *Then there exists a solution to* (1), (3), *and* (4).

Proof. Examine the following auxiliary problem:

Find a function $u(x,t) \in K_u$ *that is a regular solution to the equation*

$$Lu \equiv \frac{\partial^3 u}{\partial x^3} - \frac{\partial u}{\partial t} = 0 \tag{22}$$

in the domain $\Omega = \{(x,t) : 0 < x < 1, \; 0 < t \le T\}$ *and satisfies the conditions*

$$u(x,0) = \tau(x), \tag{23}$$

$$u(0,t) = \varphi_1(t), \quad u_x(0,t) = \varphi_2(t), \quad u_x(1,t) = \psi(t). \tag{24}$$

The method for constructing the Green's function for (22)–(24) was developed in [4]. If we apply it to (22)–(24) then, taking (9)–(13) into account, we obtain a solution to the problem in the form

$$\pi u(x,t) = -\int_0^t G_\xi(x-1;t-\tau)\psi(\tau)\,d\tau - \int_0^t G_{\xi\xi}(x-0;t-\tau)\varphi_1(\tau)\,d\tau$$
$$+\int_0^t G_\xi(x-0;t-\tau)\varphi_2(\tau)\,d\tau + \int_0^1 G(x-\xi;t-0)\tau(\xi)\,d\xi, \tag{25}$$

where $G(x-\xi;t-\tau) = U(x-\xi;t-\tau) - W(x-\xi;t-\tau)$. Here $W(x-\xi;t-\tau)$ is a solution to the problem

$$M(W) \equiv -\frac{\partial^3 W}{\partial x^3} - \frac{\partial W}{\partial t} = 0,$$
$$U|_{\xi=1} = W|_{\xi=1}, \quad U_{\xi\xi}|_{\xi=1} = W_{\xi\xi}|_{\xi=1}, \quad U|_{\xi=0} = W|_{\xi=0},$$
$$W|_{\tau=T} = 0.$$

Put $u(x,T) = \alpha(x)$. Passing to the limit as $t \to T$ and involving (25), we have

$$\pi\alpha(x) = -\int_0^T G_\xi(x-1;T-\tau)\psi(\tau)\,d\tau - \int_0^T G_{\xi\xi}(x-0;T-\tau)\varphi_1(\tau)\,d\tau$$
$$+\int_0^T G_\xi(x-0;T-\tau)\varphi_2(\tau)\,d\tau + \mu\int_0^1 \{G(x-\xi;T-0)\alpha(\xi)\,d\xi. \tag{26}$$

So, we have obtained a Fredholm integral equation of the second kind where the unknown function is $\alpha(x)$; i.e.,

$$\alpha(x) = \int_0^1 K(x,\xi)\alpha(\xi)d\xi + F(x), \tag{27}$$

where

$$K(x,\xi) \equiv \mu G(x-\xi;T-0),$$
$$F(x) \equiv -\int_0^T G_\xi(x-1;T-\tau)\psi(\tau)\,d\tau - \int_0^T G_{\xi\xi}(x-0;T-\tau)\varphi_1(\tau)\,d\tau$$
$$+\int_0^T G_\xi(x-0;T-\tau)\varphi_2(\tau)\,d\tau.$$

In view of (14)–(16), we can easily justify the following relations for $K(x,\xi)$ and $F(x)$:

$$|K(x,\xi)| < \frac{C}{|x-\xi|^{\frac{1}{4}}}, \quad F(x) \in C^3([0,1]).$$

By the uniqueness of solutions to (1), (3), (4), the integral equation (27) has the unique solution.

Similarly, we can prove the following existence theorem for Problem II.

Theorem 3. *Assume that* $0 \le \mu^2 \le \exp\{-T\}$, $\psi(t) \in C^1([0,T])$, $\varphi_2(t) \in C^2([0,T])$, *and* $\varphi_1(t) \in C^2([0,T])$. *Then there exits a solution to* (1), (5), (6).

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ON SOLVABILITY OF SOME CONJUGATION PROBLEMS FOR ELLIPTIC EQUATIONS

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Abstract. We study a general boundary value problem for a linear elliptic equation. Existence and uniqueness theorems are proven under the corresponding boundary conditions and the conjugate conditions on the interface between two media.

Keywords: boundary value problem, conjugate condition, elliptic equation

Assume that Ω is a bounded domain of the space $\mathbb{R}^n$ with smooth (for simplicity, infinitely differentiable) boundary Γ, $Q = \Omega \times (-1,1)$, $Q^- = \Omega \times (-1,0)$, and $Q^+ = \Omega \times (0,1)$. We assume next that $p(x,y)$, $c(x,y)$, $f(x,y)$, $\alpha_i(x)$, and $\beta_i(x)$ ($i = \overline{(1,4)}$) are given functions defined for $x \in \overline{\Omega}$, $y \in [-1,1]$, the function $p(x,y)$ is strictly positive for $(x,y) \in \overline{Q}$ and can have discontinuity of the first kind on the plane $y = 0$, $(\alpha_1(x), \alpha_2(x), \alpha_3(x), \alpha_4(x))$ and $(\beta_1(x), \beta_2(x), \beta_3(x), \beta_4(x))$ are some linearly independent vector-functions for every fixed $x \in \overline{\Omega}$, while B_1 and B_2 are linear operators taking $u(x,y)$ into $(B_i u)(x)$ (its properties are described below). Let L be a differential operator such that, for a given $v(x,y)$,

$$Lv \equiv \Delta_x v + \frac{\partial}{\partial y}(p(x,y)v_y) + c(x,y)v,$$

where

$$\Delta_x = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

The Conjugation Problem. *Find a solution $u(x,y)$ to the equation*

$$Lu = f(x,y) \tag{1}$$

in the cylinders Q^- and Q^+ satisfying the boundary condition

$$u(x,y)|_S = 0 \tag{2}$$

on the lateral boundary, the boundary conditions

$$u(x,-1) = 0, \quad u(x,1) = 0, \quad x \in \Omega, \tag{3}$$

on the bases, and also the conjugate conditions

$$\alpha_1(x)u(x,-0) + \alpha_2(x)u(x,+0) + \alpha_3(x)u_y(x,-0) + \alpha_4(x)u_y(x,+0) + B_1 u = 0, \tag{4}$$

$$\beta_1(x)u(x,-0) + \beta_2(x)u(x,+0) + \beta_3(x)u_y(x,-0) + \beta_4(x)u_y(x,+0) + B_2 u = 0 \tag{5}$$

on the interface between Q^- and Q^+, where $x \in \Omega$.

The problem (1)–(5) in a particular case coincides with the well-studied classical diffraction problem (see, for instance, [1–3]). This problem was not studied earlier in the general form as in our article. Note that the conjugation (diffraction) problems arise in mathematical models of many processes in physics, mechanics, biology, etc., where a process takes place in two or more adjacent substances with different physical characteristics (from the last articles we point out only [4, 5]).

The linear independence condition for the vectors $(\alpha_i(x))$ and $(\beta_i(x))$ $(i = (\overline{1,4}))$ means that at every point x in $\overline{\Omega}$ one of the second order minors of the matrix

$$\begin{pmatrix} \alpha_1(x) & \alpha_2(x) & \alpha_3(x) & \alpha_4(x) \\ \beta_1(x) & \beta_2(x) & \beta_3(x) & \beta_4(x) \end{pmatrix}$$

is different from zero.

Below we assume that one of the following minors does not vanish:

$$\Delta_1(x) = \alpha_3(x)\beta_4(x) - \alpha_4(x)\beta_3(x), \quad \Delta_2(x) = \alpha_1(x)\beta_2(x) - \alpha_2(x)\beta_1(x),$$

$$\Delta_3(x) = \alpha_1(x)\beta_4(x) - \alpha_4(x)\beta_1(x), \quad \Delta_4(x) = \alpha_2(x)\beta_3(x) - \alpha_3(x)\beta_2(x).$$

Moreover, we suppose the following:

Condition A. *If there exists a point x_0 in Ω such that $\Delta_i(x_0) \neq 0$ for one of the numbers $i = 1, 2, 3, 4$ then $\Delta_i(x) \neq 0$ for all x in $\overline{\Omega}$ and the same number i.*

Under Condition A the conjugation problem (1)–(5) can be stated as one of the following problems.

Problem I. *Find a solution $u(x, y)$ to (1) in Q^- and Q^+ such that (2) and (3) hold and*

$$u_y(x, -0) = a_1(x)u(x, -0) + a_2(x)u(x, +0) + a_3(x)B_1u + a_4(x)B_2u, \quad x \in \Omega,$$

$$u_y(x, +0) = b_1(x)u(x, -0) + b_2(x)u(x, +0) + b_3(x)B_1u + b_4(x)B_2u, \quad x \in \Omega.$$

Problem II. *Find a solution $u(x, y)$ to (1) in Q^- and Q^+ such that (2) and (3) hold and*

$$u(x, -0) = a_1(x)u_y(x, -0) + a_2(x)u_y(x, +0) + a_3(x)B_1u + a_4(x)B_2u, \quad x \in \Omega,$$

$$u(x, +0) = b_1(x)u_y(x, -0) + b_2(x)u_y(x, +0) + b_3(x)B_1u + b_4(x)B_2u, \quad x \in \Omega.$$

Problem III. *Find a solution $u(x, y)$ to (1) in Q^- and Q^+ such that (2) and (3) hold and*

$$u(x, -0) = a_1(x)u(x, +0) + a_2(x)u_y(x, -0) + a_3(x)B_1u + a_4(x)B_2u, \quad x \in \Omega,$$

$$u_y(x, +0) = b_1(x)u(x, +0) + b_2(x)u_y(x, -0) + b_3(x)B_1u + b_4(x)B_2u, \quad x \in \Omega.$$

Problem IV. *Find a solution $u(x, y)$ to (1) in Q^- and Q^+ such that (2) and (3) hold and*

$$u(x, +0) = a_1(x)u(x, -0) + a_2(x)u_y(x, +0) + a_3(x)B_1u + a_4(x)B_2u, \quad x \in \Omega,$$

$$u_y(x, -0) = b_1(x)u(x, -0) + b_2(x)u_y(x, +0) + b_3(x)B_1u + b_4(x)B_2u, \quad x \in \Omega.$$

Note that $a_1(x)$, $a_2(x)$, $a_3(x)$, $a_4(x)$, $b_1(x)$, $b_2(x)$, $b_3(x)$, and $b_4(x)$ here are calculated directly although the initial functions $\alpha_i(x), \beta_i(x)$, $i = \overline{1,4}$. Obviously, Problems III and IV are of the same type. So we can examine only one of these problems, namely, Problem III. Moreover, we assume in these problems that the function $a_1(x)b_2(x) - a_2(x)b_1(x)$ can vanish and even identically for $x \in \overline{\Omega}$. Finally, we define the function space required. Denote by V the set of functions

$$V = \{v(x,y) : v(x,y) \in W_2^2(Q^-) \cup W_2^2(Q^+)\}.$$

Before stating Theorem 1, introduce some additional notations. Put

$$\bar a_3 = \max_{\overline{\Omega}}(p(x,+0)|a_3(x)|), \quad \bar a_4 = \max_{\overline{\Omega}}(p(x,-0)|a_4(x)|),$$

$$\bar b_3 = \max_{\overline{\Omega}}(p(x,+0)|b_3(x)|), \quad \bar b_4 = \max_{\overline{\Omega}}(p(x,-0)|b_4(x)|),$$

$$A_1 = \frac{\gamma_0(m_{11}\bar b_3 + m_{12}\bar b_4) + m_{11}\bar a_3 + m_{12}\bar a_4}{2\delta_0^2} + \frac{\delta_0^2\gamma_0(\bar b_3 + \bar b_4)}{2},$$

$$A_2 = \frac{\gamma_0(m_{21}\bar b_3 + m_{22}\bar b_4) + m_{21}\bar a_3 + m_{22}\bar a_4}{2\delta_0^2} + \frac{\delta_0^2(\bar a_3 + \bar a_4)}{2},$$

$$A_3 = \frac{\gamma_0(m_{31}\bar b_3 + m_{32}\bar b_4) + m_{31}\bar a_3 + m_{32}\bar a_4}{2},$$

$$A_4 = \frac{\gamma_0(m_{41}\bar b_3 + m_{42}\bar b_4) + m_{41}\bar a_3 + m_{42}\bar a_4}{2},$$

$$\Phi(\xi,\eta) = [\gamma_0 b_2(x)p(x,+0) - A_1]\xi^2 + [\gamma_0 b_1(x)p(x,+0) - a_2(x)p(x,-0)]\xi\eta$$
$$-[a_1(x)p(x,-0) + A_2]\eta^2$$

(here γ_0 and δ_0 are some positive constants whose exact values are specified below).

Let us discuss the uniqueness questions for solutions to Problems I–III.

Theorem 1. *Assume that*

$$p(x,y) \in C^1(\overline{Q^-}), \quad p(x,y) \ge p_1 > 0, \quad (x,y) \in \overline{Q^-}, \tag{6}$$

$$p(x,y) \in C^1(\overline{Q^+}), \quad p(x,y) \ge p_2 > 0, \quad (x,y) \in \overline{Q^+}, \tag{7}$$

$$\|B_i v\|^2_{L_2(\Omega)} \le m_{1i}\|v(x,+0)\|^2_{L_2(\Omega)} + m_{2i}\|v(x,-0)\|^2_{L_2(\Omega)}$$
$$+m_{3i}\|v\|^2_{L_2(Q^-)} + m_{4i}\|v\|^2_{L_2(Q^+)}, \quad v(x,y) \in V, \ i = 1,2, \tag{8}$$

$$c(x,y) \le -c_0 \le 0, \quad (x,y) \in \overline{Q}^{\pm}, \tag{9}$$

$$\exists \gamma_0 > 0\, \exists \delta_0 > 0 : \Phi(\xi,\eta) \ge 0, \quad (\eta,\xi) \in \mathbb{R}^2,$$
$$c(x,y) + A_3/\delta_0^2 \le 0, \quad c(x,y) + A_4/\delta_0^2 \le 0, \quad x \in \overline{\Omega}, \ y \in [-1,1]. \tag{10}$$

Then Problem I *has at most one solution in* V.

Proof. It suffices to demonstrate that if the right-hand side of (1) is identically zero then so is the solution; i.e., $u(x,y) \equiv 0$.

Consider the equality

$$-\int\limits_{Q^-} uLu\,dxdy - \gamma_0 \int\limits_{Q^+} uLu\,dxdy = 0$$

and extract nonnegative summands.

Integrating by parts, we infer

$$-\int\limits_{Q^-} uLu\,dxdy - \gamma_0 \int\limits_{Q^+} uLu\,dxdy$$
$$= \sum_{i=1}^{n} \int\limits_{Q^-} u_{x_i}^2\,dxdy + \gamma_0 \sum_{i=1}^{n} \int\limits_{Q^+} u_{x_i}^2\,dxdy - \int\limits_{Q^-} c(x,y)u^2\,dxdy$$
$$-\gamma_0 \int\limits_{Q^+} c(x,y)u^2\,dxdy + \int\limits_{Q^-} p(x,y)u_y^2\,dxdy + \gamma_0 \int\limits_{Q^+} p(x,y)u_y^2\,dxdy$$
$$+ \int\limits_{\partial Q^-} p(x,y)u \cdot u_y\nu_y\,dS + \gamma_0 \int\limits_{\partial Q^+} p(x,y)u \cdot u_y\nu_y\,dS. \tag{11}$$

Transform the two last summands of (11) on using the conjugate condition as follows:

$$\int\limits_{\partial Q^-} p(x,y)u \cdot u_y\nu_y\,dS + \gamma_0 \int\limits_{\partial Q^+} p(x,y)u \cdot u_y\nu_y\,dS$$
$$= -\int\limits_{\Omega} p(x,-0)u(x,-0)u_y(x,-0)\,dx + \gamma_0 \int\limits_{\Omega} p(x,+0)u(x,+0)u_y(x,+0)\,dx$$
$$= \int\limits_{\Omega} \gamma_0 b_2(x)p(x,+0)u^2(x,+0)\,dx$$
$$+ \int\limits_{\Omega} [\gamma_0 b_1(x)p(x,+0) - a_2(x)p(x,-0)]u(x,+0)u(x,-0)\,dx$$
$$- \int\limits_{\Omega} p(x,-0)a_1(x)u^2(x,-0)]\,dx$$
$$+ \int\limits_{\Omega} [\gamma_0 b_3(x)p(x,+0)u(x,+0) - a_3(x)p(x,-0)u(x,-0)]B_1u\,dx$$
$$+ \int\limits_{\Omega} [\gamma_0 b_4(x)p(x,+0)u(x,+0) - a_4(x)p(x,-0)u(x,-0)]B_2u\,dx.$$

Employing the above notations and (8), we see that

$$\sum_{i=1}^{n} \int_{Q^-} u_{x_i}^2\,dxdy + \gamma_0 \sum_{i=1}^{n} \int_{Q^+} u_{x_i}^2\,dxdy$$
$$+ \int\limits_{Q^-} p(x,y)u_y^2\,dxdy + \gamma_0 \int\limits_{Q^+} p(x,y)u_y^2\,dxdy + \int\limits_{\Omega} [\gamma_0 b_2(x)p(x,+0) + A_1]u^2(x,+0)\,dx$$
$$+ \int\limits_{\Omega} [\gamma_0 b_1(x)p(x,+0) - a_2(x)p(x,-0)]u(x,+0)u(x,-0)\,dx$$
$$- \int\limits_{\Omega} [p(x,-0)a_1(x) - A_2]u^2(x,-0)\,dx - \int\limits_{Q^-} [c(x,y) - A_3/\delta_0^2]u^2(x,y)\,dxdy$$
$$-\gamma_0 \int\limits_{Q^+} [c(x,y) - A_4/\delta_0^2]u^2(x,y)\,dxdy \leq 0.$$

Hence,

$$\sum_{i=1}^{n}\int_{Q^-} u_{x_i}^2\,dxdy+\gamma_0\sum_{i=1}^{n}\int_{Q^+} u_{x_i}^2\,dxdy+\int\limits_{Q^-} p(x,y)u_y^2\,dxdy+\gamma_0\int\limits_{Q^+} p(x,y)u_y^2\,dxdy$$

$$-\int\limits_{Q^-} c(x,y)u^2\,dxdy-\gamma_0\int\limits_{Q^+} c(x,y)u^2\,dxdy\le 0. \tag{12}$$

As an obvious result, $u(x,y)\equiv 0$ in Q. The theorem is proven.

To state Theorem 2, we introduce the notations:

$$A_{11}=\frac{\gamma_0(\bar b_3+\bar b_4)}{2},\quad A_{21}=\frac{\bar a_3+\bar a_4}{2},$$

$$A_5=\frac{\gamma_0(m_{11}\bar b_3+m_{12}\bar b_4)+m_{11}\bar a_3+m_{12}\bar a_4}{2},$$

$$A_6=\frac{\gamma_0(m_{21}\bar b_3+m_{22}\bar b_4)+m_{21}\bar a_3+m_{22}\bar a_4}{2},$$

$$\Phi_1(\xi,\eta)=\big[\gamma_0 b_2(x)p(x,+0)-A_{11}\delta_1^2\big]\xi^2+[\gamma_0 b_1(x)p(x,+0)-a_2(x)p(x,-0)]\xi\eta$$
$$-\big[a_1(x)p(x,-0)+A_{21}\delta_1^2\big]\eta^2.$$

Theorem 2. *Assume the conditions* (6)–(9) *as well as the conditions*

$$\exists\gamma_0>0\,\exists\delta_1>0:\Phi_1(\eta,\xi)\ge 0,\quad (\eta,\xi)\in\mathbb{R}^2,$$
$$p_1-A_5/\delta_1^2>0,\quad p_2-A_6/\delta_1^2>0,$$
$$c(x,y)+A_3/\delta_1^2\le 0,\quad c(x,y)+A_4/\delta_1^2\le 0,\quad x\in\overline{\Omega},\ y\in[-1,1].$$

Then Problem II *has at most one solution in* V*.*

Proof. As in the proof of Theorem 1, it suffices to prove that $u(x,y)\equiv 0$ whenever the right-hand side of (1) is identically zero. The conjugate conditions, (8), and the obvious inequalities

$$\int\limits_{\Omega} u^2(x,-0)\,dx\le\int\limits_{Q^-} u_y^2(x,y)\,dxdy,\quad \int\limits_{\Omega} u^2(x,+0)\,dx\le\int\limits_{Q^+} u_y^2(x,y)\,dxdy \tag{13}$$

imply that

$$\sum_{i=1}^{n}\int_{Q^-} u_{x_i}^2\,dxdy+\gamma_0\sum_{i=1}^{n}\int_{Q^+} u_{x_i}^2\,dxdy$$
$$+\int\limits_{Q^-}\big[p(x,y)+A_5/\delta_1^2\big]u_y^2\,dxdy+\gamma_0\int\limits_{Q^+}\big[p(x,y)+A_6/\delta_1^2\big]u_y^2\,dxdy$$
$$+\int\limits_{\Omega}\big[\gamma_0 b_2(x)p(x,+0)+A_{11}\delta_1^2\big]u_y^2(x,+0)\,dx$$
$$-\int\limits_{\Omega}\big[p(x,-0)a_1(x)+A_{21}\delta_1^2\big]u_y^2(x,-0)\,dx$$
$$-\int\limits_{\Omega}[\gamma_0 b_1(x)p(x,+0)-a_2(x)p(x,-0)]u(x,+0)u_y(x,-0)\,dx$$
$$-\int\limits_{Q^-}\big[c(x,y)+A_3/\delta_1^2\big]u^2(x,y)\,dxdy-\gamma_0\int\limits_{Q^+}\big[c(x,y)+A_4/\delta_1^2\big]u^2(x,y)\,dxdy\le 0.$$

Hence, (12) holds and so $u(x,y)\equiv 0$ in Q. The theorem is proven.

Put

$$\Phi_2(\xi,\eta)=\big[\gamma_0 b_2(x)p(x,+0)-A_{11}\delta_2^2\big]\xi^2+[\gamma_0 b_1(x)p(x,+0)-a_2(x)p(x,-0)]\xi\eta \\ -\big[a_1(x)p(x,-0)+A_{21}\delta_2^2\big]\eta^2.$$

Theorem 3. *Assume the conditions* (6)–(9) *as well as the conditions*

$$\exists\gamma_0>0\,\exists\delta_2>0:\Phi_2(\eta,\xi)\geq 0,\quad (\eta,\xi)\in\mathbb{R}^2,$$

$$p_1-A_5/\delta_2^2>0,\quad p_2-A_6/\delta_2^2>0,$$

$$c(x,y)+A_3/\delta_2^2\leq 0,\quad c(x,y)+A_4/\delta_2^2\leq 0,\quad x\in\overline{\Omega},\ y\in[-1,1].$$

Then Problem III *has at most one solution in* V.

Proof. Arguing as in the proof of Theorem 1, taking (8) into account, we see that

$$\sum_{i=1}^n\int_{Q^-}u_{x_i}^2\,dxdy+\gamma_0\sum_{i=1}^n\int_{Q^+}u_{x_i}^2\,dxdy$$
$$+\int\limits_{Q^-}\big[p(x,y)+A_5/\delta_2^2\big]u_y^2\,dxdy+\gamma_0\int\limits_{Q^+}\big[p(x,y)+A_6/\delta_2^2\big]u_y^2\,dxdy$$
$$+\int\limits_{\Omega}\big[\gamma_0 b_2(x)p(x,+0)+A_{11}\delta_2^2\big]u_y^2(x,+0)\,dx$$
$$-\int\limits_{\Omega}\big[p(x,-0)a_1(x)+A_{21}\delta_2^2\big]u_y^2(x,-0)\,dx$$
$$-\int\limits_{\Omega}\big[\gamma_0 b_1(x)p(x,+0)-a_2(x)p(x,-0)\big]u(x,+0)u_y(x,-0)\,dx$$
$$-\int\limits_{Q^-}\big[c(x,y)+A_3/\delta_2^2\big]u^2(x,y)\,dxdy-\gamma_0\int\limits_{Q^+}\big[c(x,y)+A_4/\delta_2^2\big]u^2(x,y)\,dxdy\leq 0.$$

Therefore, (12) is fulfilled and so $u(x,y)\equiv 0$ in Q. The theorem is proven.

Before stating the next theorem, we introduce the new notations:

$$\tilde{a}_3=\max_{\overline{\Omega}}(|a_3(x)|),\quad \tilde{a}_4=\max_{\overline{\Omega}}(|a_4(x)|),$$
$$\tilde{b}_3=\max_{\overline{\Omega}}(|b_3(x)|),\quad \tilde{b}_4=\max_{\overline{\Omega}}(|b_4(x)|).$$

Theorem 4 (of existence). *Assume the conditions* (6)–(10) *as well as the conditions*

$$c(x,y)\in(\overline{Q}),\quad a_i(x)\in C^2(\overline{\Omega}),\ i=1,2. \tag{14}$$

Then there exist operators $\widetilde{B}_i$ *such that*

$$\frac{\partial}{\partial x_j}(B_i v)=B_i v_{x_j}+\widetilde{B}_i v,$$
$$\|\widetilde{B}_i v\|^2_{L_2(\Omega)}\leq\widetilde{m}_{1i}\|v(x,+0)\|^2_{L_2(\Omega)}+\widetilde{m}_{2i}\|v(x,-0)\|^2_{L_2(\Omega)}$$
$$+\widetilde{m}_{3i}\|v\|^2_{L_2(Q^-)}+\widetilde{m}_{4i}\|v\|^2_{L_2(Q^+)},\quad v(x,y)\in V,\ i=1,2, \tag{15}$$

for all $v(x,y) \in V$, and there exist positive numbers δ_{01}–δ_{04} such that

$$
\begin{aligned}
&2 - \big(1+\tilde a_3^2\big)\delta_{01}^2 - \big(1+\tilde a_4^2\big)\delta_{02}^2 - \widetilde m_{21}\left(\frac{\tilde a_3^2}{\delta_{01}^2} + \frac{\gamma_0 \tilde b_3^2}{\delta_{03}^2}\right) - \widetilde m_{32}\left(\frac{\tilde a_4^2}{\delta_{02}^2} + \frac{\gamma_0 \tilde b_4^2}{\delta_{04}}\right) > 0,\\
&2 - \big(1+\tilde b_3^2\big)\delta_{03}^2 - \big(1+\tilde b_4^2\big)\delta_{04}^2 - \widetilde m_{12}\left(\frac{\tilde a_4^2}{\delta_{02}^2} + \frac{\gamma_0 \tilde b_4^2}{\delta_{04}^2}\right) - \widetilde m_{11}\left(\frac{\tilde a_3^2}{\delta_{01}^2} + \frac{\gamma_0 \tilde b_3^2}{\delta_{01}}\right) > 0.
\end{aligned}
\tag{16}
$$

Then, for every function $f(x,y)$ from $L_2(Q)$, there exists a solution $u(x,y)$ in V to Problem I.

Proof. We apply the method of continuation in a parameter. Let λ be a number in $[0,1]$. Consider the following family of boundary value problems:

Problem I$_\lambda$. *Find a solution $u(x,y)$ to* (1) *in Q^- and Q^+ such that* (2) *and* (3) *hold and*

$$
\begin{aligned}
u_y(x,-0) &= \lambda[a_1(x)u(x,-0) + a_2(x)u(x,+0) + a_3(x)B_1u + a_4(x)B_2u],\\
u_y(x,+0) &= \lambda[b_1(x)u(x,-0) + b_2(x)u(x,+0) + b_3(x)B_1u + b_4(x)B_2u].
\end{aligned}
\tag{17}
$$

Denote by Λ the set of numbers λ from the segment $[0,1]$ for which Problem I$_\lambda$ is solvable in V. In accord with the theorem about the method of continuation in a parameter [6], Λ coincides with $[0,1]$ if Λ is nonempty, open, and closed (in the topology of the metric space $X=[0,1]$). The coincidence of Λ with $[0,1]$ means that Problem I$_1$, i.e., the initial problem I, is solvable in V.

Since 0 belongs to Λ [7], Λ is nonempty.

The set Λ is clopen if all solutions to Problem I$_\lambda$ from V satisfy the a priori estimate

$$\|u\|_V \le R_0$$

with a constant K independent of $u(x,y)$ and λ. Show that this estimate is valid.

Consider the equality

$$- \int_{Q^-} uLu\,dxdy - \gamma_0 \int_{Q^+} uLu\,dxdy = - \int_{Q^-} uf(x,y)\,dxdy - \gamma_0 \int_{Q^+} uf(x,y)\,dxdy. \tag{18}$$

Integrating by parts, we have

$$
\begin{aligned}
&\sum_{i=0}^{n} \int_{Q^-} u_{x_i}^2\,dxdy + \gamma_0 \sum_{i=0}^{n} \int_{Q^+} u_{x_i}^2\,dxdy - \int_{Q^-} c(x,y)u^2\,dxdy - \gamma_0 \int_{Q^+} c(x,y)u^2\,dxdy\\
&\quad + \int_{Q^-} p(x,y)u_y^2\,dxdy + \gamma_0 \int_{Q^+} p(x,y)u_y^2\,dxdy + \int_{\partial Q^-} p(x,y)u\cdot u_y \nu_y\,dS\\
&\quad + \gamma_0 \int_{\partial Q^+} p(x,y)u\cdot u_y\nu_y\,dS = - \int_{Q^-} uf(x,y)\,dxdy - \gamma_0 \int_{Q^+} uf(x,y)\,dxdy.
\end{aligned}
$$

Taking the inequality

$$\int_{Q^-} u^2(x,y)\,dxdy \le \int_{Q^-} u_y^2(x,y)\,dxdy, \qquad \int_{Q^+} u^2(x,y)\,dxdy \le \int_{Q^+} u_y^2(x,y)\,dxdy$$

into account, applying to the right-hand side the Young inequality, and using (16) of Problem I_λ and the conditions of the theorem, we see that

$$\sum_{i=1}^{n}\int_{Q^-} u_{x_i}^2\,dxdy+\gamma_0\sum_{i=1}^{n}\int_{Q^+} u_{x_i}^2\,dxdy+\int\limits_{Q^-} p(x,y)u_y^2\,dxdy$$
$$+\gamma_0\int\limits_{Q^+} p(x,y)u_y^2\,dxdy+\lambda\int\limits_{\Omega}[\gamma_0 b_2(x)p(x,+0)+A_1]u^2(x,+0)\,dx$$
$$+\lambda\int\limits_{\Omega}[\gamma_0 b_1(x)p(x,+0)-a_2(x)p(x,-0)]u(x,+0)u(x,-0)\,dx$$
$$-\lambda\int\limits_{\Omega}[p(x,-0)a_1(x)-A_2]u^2(x,-0)\,dx$$
$$-\int\limits_{Q^-}\left[c(x,y)+A_3/\delta_0^2\right]u^2(x,y)\,dxdy-\gamma_0\int\limits_{Q^+}\left[c(x,y)+A_4/\delta_0^2\right]u^2(x,y)\,dxdy$$
$$\le\frac{1}{2\delta^2}\int\limits_{Q^-} f^2\,dxdy+\frac{\gamma_0}{2\delta^2}\int\limits_{Q^+} f^2\,dxdy+\frac{\delta^2}{2}\int\limits_{Q^-} u_y^2\,dxdy+\frac{\gamma_0\delta^2}{2}\int\limits_{Q^+} u_y^2\,dxdy.$$

From (6)–(9) it follows that

$$k\left\{\sum_{i=1}^{n}\int\limits_{Q^-} u_{x_i}^2\,dxdy+\int\limits_{Q^-} u_y^2\,dxdy+\sum_{i=1}^{n}\int\limits_{Q^+} u_{x_i}^2\,dxdy+\int\limits_{Q^+} u_y^2\,dxdy\right\}$$
$$\le K\left(\int\limits_{Q^-} f^2(x,y)\,dxdy+\int\limits_{Q^+} f^2(x,y)\,dxdy\right),$$

where $k=\min\left(1,\frac{p_0}{2},\gamma,\frac{\gamma_0 p_0}{2}\right)>0$ and $K=\max\left(\frac{1}{2p_0},\frac{\gamma_0}{2p_0}\right)$.

Hence, the first estimate is of the form

$$\|u\|^2_{W_2^1(Q^-)}+\|u\|^2_{W_2^1(Q^+)}\le K_1\left(\int\limits_{Q^-} f^2(x,y)\,dxdy+\int\limits_{Q^+} f^2(x,y)\,dxdy\right). \tag{19}$$

To establish the second a priori estimate, we examine the equality

$$\int\limits_{Q^-} u_{yy}Lu\,dxdy+\gamma_0\int\limits_{Q^+} u_{yy}Lu\,dxdy=\int\limits_{Q^-} u_{yy}f(x,y)\,dxdy+\gamma_0\int\limits_{Q^+} u_{yy}f(x,y)\,dxdy. \tag{20}$$

Integrating by parts, we infer that

$$\sum_{i=1}^{n}\int\limits_{Q^-} u_{x_iy}^2\,dxdy+\gamma_0\sum_{i=1}^{n}\int\limits_{Q^+} u_{x_iy}^2\,dxdy-\sum_{i=1}^{n}\int\limits_{\Omega} u_{x_iy}(x,-0)u_{x_i}(x,-0)\,dx$$
$$+\gamma_0\sum_{i=1}^{n}\int\limits_{\Omega} u_{x_iy}(x,+0)u_{x_i}(x,+0)\,dx+\int\limits_{Q^-} pu_{yy}^2\,dxdy+\gamma_0\int\limits_{Q^+} pu_{yy}^2\,dxdy$$
$$=\int\limits_{Q^-} f(x,y)u_{yy}\,dxdy+\gamma_0\int\limits_{Q^+} f(x,y)u_{yy}\,dxdy-\int\limits_{Q^-} p_y u_y u_{yy}\,dxdy$$

$$-\gamma_0 \int\limits_{Q^+} p_y u_y u_{yy}\, dxdy - \int\limits_{Q^-} c(x,y) u u_{yy}\, dxdy - \gamma_0 \int\limits_{Q^+} c(x,y) u u_{yy}\, dxdy. \qquad (21)$$

Transform the third and forth summands of the right-hand side of (21) with the use of (17) as follows:

$$- \int\limits_{\Omega} u_{x_i y}(x,-0) u_{x_i}(x,-0)\, dx + \gamma_0 \int\limits_{\Omega} u_{x_i y}(x,+0) u_{x_i}(x,+0)\, dx$$

$$= \lambda \int\limits_{\Omega} \big[\gamma_0 b_2(x) u_{x_i}^2(x,+0) + [\gamma_0 b_1(x) - a_2(x)] u_{x_i}(x,-0) u_{x_i}(x,+0)$$

$$-a_1(x) u_{x_i}^2(x,-0) - (u_{x_i}(x,-0) u(x,-0) a_{1x_i}(x) - u_{x_i}(x,-0) u(x,+0) a_{2x_i}(x)\big]\, dx$$

$$-\lambda \int\limits_{\Omega} \Big(u_{x_i}(x,-0) \frac{\partial}{\partial x_i}(B_1 u) a_3(x) - u_{x_i}(x,-0) B_1 u a_{3x_i}(x)$$

$$-u_{x_i}(x,-0) \frac{\partial}{\partial x_i}(B_2 u) a_4(x) - u_{x_i}(x,-0) B_2 u a_{4x_i}(x)$$

$$+\gamma_0 \Big(u_{x_i}(x,+0) u(x,-0) b_{1x_i}(x) + u_{x_i}(x,+0) u(x,+0) b_{2x_i}(x)$$

$$+u_{x_i}(x,+0) \frac{\partial}{\partial x_i}(B_1 u) b_3(x) + u_{x_i}(x,+0) B_1 u b_{3x_i}(x)$$

$$+u_{x_i}(x,+0) \frac{\partial}{\partial x_i}(B_2 u) b_4(x) + u_{x_i}(x,+0) B_2 u b_{4x_i}(x) \Big)\Big)\, dx.$$

All but first summands on the right-hand side are estimated by the Young inequality, inequality (13) applied to the derivatives $u_{x_i}(x,-0)$ and $u_{x_i}(x,+0)$, and the conditions (8) and (15) on B_1 and B_2. Next, the summands on the right-hand side of (20) are estimated by the Young inequality and the first estimate. In result, we find that

$$\int\limits_{\Omega} \big(\gamma_0 u_{x_i}^2(x,+0) b_2(x) + u_{x_i}(x,-0) u_{x_i}(x,+0)[\gamma_0 b_1(x) - a_2(x)] - u_{x_i}^2(x,-0) a_1(x)\big)\, dx$$

$$+\frac{p_1}{2} \int\limits_{Q^-} u_{yy}^2\, dxdy + \frac{\gamma_0 p_2}{2} \int\limits_{Q^+} u_{yy}^2\, dxdy + \sum_{i=1}^{n} \int\limits_{Q^-} u_{x_i y}^2 \Big(1 - \frac{(1+\tilde{a}_3^2)\delta_{01}^2}{2} - \frac{(1+\tilde{a}_4^2)\delta_{02}^2}{2}$$

$$-\widetilde{m}_{21} \left[\frac{\tilde{a}_3^2}{\delta_{01}^2} + \frac{\gamma_0 \tilde{b}_3^2}{\delta_{03}^2} \right] - \widetilde{m}_{32} \left[\frac{\tilde{a}_4^2}{\delta_{02}^2} + \frac{\gamma_0 \tilde{b}_4^2}{\delta_{04}} \right] \Big)\, dxdy + \sum_{i=1}^{n} \int\limits_{Q^+} u_{x_i y}^2 \Big(1 - \frac{(1+\tilde{b}_3^2)\delta_{03}^2}{2}$$

$$-\frac{(1+\tilde{b}_4^2)\delta_{04}^2}{2} - \widetilde{m}_{12} \left[\frac{\tilde{a}_4^2}{\delta_{02}^2} + \frac{\gamma_0 \tilde{b}_4^2}{\delta_{04}^2} \right] - \widetilde{m}_{11} \left[\frac{\tilde{a}_3^2}{\delta_{01}^2} + \frac{\gamma_0 - \tilde{b}_3^2}{\delta_{01}} \right] \Big)\, dxdy$$

$$\le \delta \left[\int\limits_{Q^-} u_{yy}^2\, dxdy + \gamma_0 \int\limits_{Q^+} u_{yy}^2\, dxdy \right] + M.$$

The condition (16) yields

$$\int\limits_{Q^-} u_{yy}^2\, dxdy + \sum_{i=1}^{n} \int\limits_{Q^-} u_{x_i y}^2\, dxdy + \gamma_0 \int\limits_{Q^+} u_{yy}^2\, dxdy + \gamma_0 \sum_{i=1}^{n} \int\limits_{Q^+} u_{x_i y}^2\, dxdy \le M_1.$$

The third a priori estimate

$$\sum_{i,j=1}^{n} \int_{Q^-} u_{x_i x_j}^2 \, dxdy + \gamma_0 \sum_{i,j=1}^{n} \int_{Q^+} u_{x_i x_j}^2 \, dxdy \le M_2$$

follows obviously from the first two estimates and the second main inequality for elliptic operators [7]. These three estimates ensure the desired estimate

$$\|u\|_V \le R_0, \tag{22}$$

which means that Problem I_λ is solvable in V for all λ in $[0,1]$; i.e. for $\lambda = 1$ as well. In this case the initial problem I is solvable in V. The theorem is proven.

REMARK. A solution $u(x,y)$ to Problem I meets the relations $u_{x_i y}(x,-0) \in L_2(\Omega)$ and $u_{x_i y}(x,+0) \in L_2(\Omega)$, $i = 1,\dots,n$. This fact is a consequence of the conjugate conditions where every summand on the right-hand side has a generalized derivative with respect to x_i.

Given x in $\overline{\Omega}$ and the variables ξ and η, define the quadratic form of the variables (ξ,η) as

$$\Phi_3(x,\xi,\eta) = -a_1(x)\xi^2 + [\gamma_1 b_1(x) - a_2(x)]\xi\eta + \gamma_1 b_2(x)\eta^2,$$

where γ_1 is a positive number whose role will be elucidated below.

Theorem 5 (of existence). *Assume the conditions* (6)–(9), (14), *and* (15) *as well as the conditions*

$$a_1(x) \le 0, \quad b_2(x) \ge 0, \quad a_1(x)b_2(x) - a_2(x)b_1(x) \le 0, \quad x \in \overline{\Omega},$$

$$\exists \gamma_1 > 0 : \Phi_3(x,\xi,\eta) \ge \varphi_1(x)\xi^2 + \varphi_2(x)\eta^2, \quad \varphi_1(x) \ge 0,\ \varphi_2(x) \ge 0,\ x \in \overline{\Omega},$$

$$|a_{2x_i}(x)| \le C\sqrt{\varphi_1(x)}, \quad |b_{1x_i}(x)| \le C\sqrt{\varphi_2(x)}, \quad C = \mathrm{const},\ x \in \overline{\Omega},\ i = 1,\dots,n.$$

Assume further that there exist positive numbers δ_{01}–δ_{04} *such that*

$$2 - \big(1+\tilde a_3^2\big)\delta_{01}^2 - \big(1+\tilde a_4^2\big)\delta_{02}^2 - \widetilde m_{21}\bigg(\frac{\tilde a_3^2}{\delta_{01}^2} + \frac{\gamma_1 \tilde b_3^2}{\delta_{03}^2}\bigg) - \widetilde m_{32}\bigg(\frac{\tilde a_4^2}{\delta_{02}^2} + \frac{\gamma_1 \tilde b_4^2}{\delta_{04}}\bigg) > 0,$$

$$2 - \big(1+\tilde b_3^2\big)\delta_{03}^2 - \big(1+\tilde b_4^2\big)\delta_{04}^2 - \widetilde m_{12}\bigg(\frac{\tilde a_4^2}{\delta_{02}^2} + \frac{\gamma_1 \tilde b_4^2}{\delta_{04}^2}\bigg) - \widetilde m_{11}\bigg(\frac{\tilde a_3^2}{\delta_{01}^2} + \frac{\gamma_1 \tilde b_3^2}{\delta_{01}}\bigg) > 0.$$

Then, for every function $f(x,y)$ from $L_2(Q)$, there exists a solution $u(x,y)$ to Problem II *in V.*

PROOF. First, we consider the case $a_j(x) \equiv 0$, $b_j(x) \equiv 0$, $j = 3,4$. Let ε be a positive number. Put

$$a_{1\varepsilon}(x) = a_1(x) - \varepsilon, \quad b_{2\varepsilon}(x) = b_2(x) + \varepsilon.$$

Examine the following problem: *Find a solution $u(x,y)$ to* (1) *in the cylinders Q^- and Q^+ satisfying* (2), (3), *and such that*

$$\begin{aligned} u(x,-0) &= a_{1\varepsilon}(x)u_y(x,-0) + a_2(x)u_y(x,+0), \quad x \in \Omega,\\ u(x,+0) &= b_1(x)u_y(x,-0) + b_{2\varepsilon}(x)u_y(x,+0), \quad x \in \Omega. \end{aligned} \tag{23}$$

Since $a_{1\varepsilon}(x)b_{2\varepsilon}(x) - a_2(x)b_1(x)$ is strictly negative in $\overline{\Omega}$, from (22) and (23) it follows that

$$\begin{aligned} u_y(x,-0) &= \tilde a_{1\varepsilon}(x)u(x,-0) + \tilde a_{2\varepsilon}(x)u(x,+0), \quad x \in \Omega,\\ u_y(x,+0) &= \tilde b_{2\varepsilon}(x)u(x,-0) + \tilde b_{2\varepsilon}(x)u(x,+0), \quad x \in \Omega, \end{aligned} \tag{24}$$

i.e., we arrive at the conditions of Problem I. In accord with Theorem 4 the problem in question has a solution $u^\varepsilon(x,y)$ in V. Demonstrate that the family $\{u^\varepsilon(x,y)\}$ satisfies the estimates that allow us to pass to the limit as $\varepsilon \to 0$. We omit the index ε here.

Solutions to (1)–(3), (23) satisfy (19) (the proof repeats verbatim the proof of the same estimate in Theorem 4). Next, transforming (20) to the form (21), we separately examine the third and forth summands. Using the conjugate conditions (23), we infer

$$-\sum_{i=1}^{n}\int\limits_{\Omega} u_{x_iy}(x,-0)u_{x_i}(x,-0)\,dx + \gamma_1\sum_{i=1}^{n}\int\limits_{\Omega} u_{x_iy}(x,+0)u_{x_i}(x,+0)\,dx$$

$$= \sum_{i=1}^{n}\int\limits_{\Omega}\big[\gamma_1 b_2(x)u^2_{x_i,y}(x,+0) + [\gamma_1 b_1(x) - a_2(x)]u_{x_i,y}(x,-0)u_{x_i,y}(x,+0)$$

$$-a_1(x)u^2_{x_i,y}(x,-0)\big]\,dx + \int\limits_{\Omega}\bigg[\frac{1}{2}\Delta a_1(x)u_y^2(x,-0) - \frac{1}{2}\gamma_1\Delta b_2(x)u_y^2(x,+0)$$

$$-\sum_{i=1}^{n} a_{2x_i}(x)u_y(x,+0)u_{x_iy}(x,-0) + \gamma_1\sum_{i=1}^{n} b_{1x_i}(x)u_y(x,-0)u_{x_iy}(x,+0)$$

$$+\varepsilon\sum_{i=1}^{n}\big(u^2_{x_iy}(x,-0) + \gamma_1 u^2_{x_iy}(x,+0)\big)\bigg]\,dx.$$

In result, we arrive at the equality

$$\sum_{i=1}^{n}\int\limits_{\Omega}\big[b_2(x)\gamma_1 u^2_{x_iy}(x,+0) + [\gamma_1 b_1(x) - a_2(x)]u_{x_iy}(x,-0)u_{x_iy}(x,+0)$$

$$-a_1(x)u^2_{x_iy}(x,-0)\big]\,dx + \frac{p_1}{2}\int\limits_{Q^-} u^2_{yy}\,dxdy + \frac{\gamma_1 p_2}{2}\int\limits_{Q^+} u^2_{yy}\,dxdy$$

$$+\sum_{i=1}^{n}\int\limits_{Q^-} u^2_{x_iy}\,dxdy + \gamma_1\sum_{i=1}^{n}\int\limits_{Q^+} u^2_{x_iy}\,dxdy + \varepsilon\sum_{i=1}^{n}\int\limits_{\Omega}\big[u^2_{x_iy}(x,-0) + \gamma_1 u^2_{x_iy}(x,+0)\big]\,dx$$

$$= \int\limits_{\Omega}\bigg[-\frac{1}{2}\Delta_x a_1(x)u_y^2(x,-0) + \frac{1}{2}\Delta_x b_1(x)u_y^2(x,+0)\bigg]\,dx$$

$$+\sum_{i=1}^{n}\int\limits_{\Omega}\big[a_{2x_i}u_y(x,+0)u_{x_iy}(x,-0) + \gamma_1 b_{1x_i}u_y(x,+0)u_{x_iy}(x,-0)\big]\,dx$$

$$+\int\limits_{Q^-} c(x,y)u_{yy}u\,dxdy - \gamma_1\int\limits_{Q^+} c(x,y)u_{yy}u\,dxdy.$$

Applying the Young inequality and using the conditions of the theorem and the integral inequalities of the form (13), we obtain the estimate

$$\int\limits_{Q^-} u^2_{yy}\,dxdy + \sum_{i=1}^{n}\int\limits_{Q^-} u^2_{x_iy}\,dxdy + \int\limits_{Q^+} u^2_{yy}\,dxdy + \sum_{i=1}^{n}\int\limits_{Q^+} u^2_{x_iy}\,dxdy$$

$$+\varepsilon\sum_{i=1}^{n}\int\limits_{\Omega}\big(u^2_{x_iy}(x,-0) + u^2_{x_iy}(x,+0)\big)\,dx \le M_3. \tag{25}$$

Now the third a priori estimate

$$\sum_{i,j=1}^{n} \int_{Q^-} u_{x_i x_j}^2 \, dxdy + \sum_{i,j=1}^{n} \int_{Q^+} u_{x_i x_j}^2 \, dxdy \le M_4 \tag{26}$$

is an obvious consequence of the first two estimates and the second main inequality for elliptic operators [7].

The estimates (19), (25), and (26), the reflexivity of Hilbert spaces, and the embedding theorems (see [7, 8]) imply that there exist a sequence of numbers $\{\varepsilon_m\}$, a sequence of functions $\{u_m(x,y)\}$ $(u_m(x,y) = u^{\varepsilon_m}(x,y))$ from the family $\{u^\varepsilon(x,y)\}$, and a function $u(x,y)$ such that $\varepsilon_m \to 0$, $u_m(x,y) \to u(x,y)$ weakly in $W_2^2(Q^-)$ and $W_2^2(Q^+)$,

$$u_m(x,-0) \to u(x,-0), \quad u_m(x,+0) \to u(x,+0), \quad u_{my}(x,-0) \to u_y(x,-0),$$

$$u_{my}(x,+0) \to u_y(x,+0), \quad \varepsilon_m u_{my}(x,-0) \to 0,$$

$$\varepsilon_m u_{my}(x,+0) \to 0 \text{ almost everywhere in } \Omega$$

as $m \to \infty$. Hence, the limit function $u(x,y)$ satisfies (1) in Q^- and Q^+, where (2), (3), and the conjugate conditions are those of Problem II.

In the case of nonzero functions $a_3(x)$, $b_3(x)$, $a_4(x)$, and $b_4(x)$ the proof is similar to the above proof (some arguments are more cumbersome). The theorem is proven.

Given $x \in \overline{\Omega}$ and variables ξ and η, define the quadratic form $\Phi_4(x,\xi,\eta)$ of variables (ξ,η) as follows:

$$\Phi_4(x,\xi,\eta) = -a_2(x)\xi^2 + [\gamma_2 b_2(x) - a_1(x)]\xi\eta + \gamma_2 b_1(x)\eta^2,$$

where γ_2 is a positive number whose role is elucidated below.

Theorem 6 (of existence). *Assume the conditions* (6)–(9), (14), (17), *as well as the conditions*

$$a_2(x) \le 0, \quad x \in \overline{\Omega},$$

$$\exists \gamma_2 > 0 : \Phi_4(x,\xi,\eta) \ge \psi_1(x)\xi^2 + \psi_2(x)\eta^2, \quad \psi_1(x) \ge 0, \ \psi_2(x) \ge 0, \ x \in \bar{\Omega},$$

$$|a_{1x_i}(x)| \le C\sqrt{\psi_1(x)}, \quad |b_{1x_i}(x)| \le C\sqrt{\psi_2(x)}, \quad C = \text{const}, \ x \in \overline{\Omega}, \ i = 1,\dots,n.$$

Assume further that there exist positive numbers δ_{01}–δ_{04} such that

$$2 - \left(1 + \tilde{a}_3^2\right)\delta_{01}^2 - \left(1 + \tilde{a}_4^2\right)\delta_{02}^2 - \widetilde{m}_{21}\left(\frac{\tilde{a}_3^2}{\delta_{01}^2} + \frac{\gamma_2 \tilde{b}_3^2}{\delta_{03}^2}\right) - \widetilde{m}_{32}\left(\frac{\tilde{a}_4^2}{\delta_{02}^2} + \frac{\gamma_2 \tilde{b}_4^2}{\delta_{04}}\right) > 0,$$

$$2 - \left(1 + \tilde{b}_3^2\right)\delta_{03}^2 - \left(1 + \tilde{b}_4^2\right)\delta_{04}^2 - \widetilde{m}_{12}\left(\frac{\tilde{a}_4^2}{\delta_{02}^2} + \frac{\gamma_2 \tilde{b}_4^2}{\delta_{04}^2}\right) - \widetilde{m}_{11}\left(\frac{\tilde{a}_3^2}{\delta_{01}^2} + \frac{\gamma_2 \tilde{b}_3^2}{\delta_{01}}\right) > 0.$$

Then, for every function $f(x,y)$ from $L_2(Q)$, there exists a solution $u(x,y)$ to Problem III *in V.*

Proof. As in the proof of Theorem 5, we study first the case of $a_j(x) \equiv 0$ and $b_j(x) \equiv 0$, $j = 3, 4$. Put

$$a_{2\varepsilon}(x) = a_2(x) - \varepsilon, \quad \varepsilon > 0.$$

Consider the problem of finding a solution to (1) such that (2) and (3) hold and

$$\begin{aligned} u(x,-0) &= a_1(x)u_y(x,+0) + a_{2\varepsilon}(x)u_y(x,-0), \quad x \in \Omega, \\ u_y(x,+0) &= b_1(x)u_y(x,+0) + b_2(x)u_y(x,-0), \quad x \in \Omega. \end{aligned} \tag{27}$$

We can pass to the relations of the form (24) which are the conditions of Problem I. This problem has a solution $u^\varepsilon(x,y)$ in V and the first a priori estimate (19) holds for $u^\varepsilon(x,y)$.

To justify further estimates, we examine (21). The conjugate condition (27) yields

$$\begin{aligned}
&\sum_{i=1}^{n}\int\limits_{\Omega}\big[\gamma_2 b_1(x)u_{x_i}^2(x,+0)+[\gamma_2 b_2(x)-a_1(x)]u_{x_iy}(x,-0)u_{x_i}(x,+0)\\
&\quad -a_2(x)u_{x_iy}^2(x,-0)\big]\,dx+\frac{p_1}{2}\int\limits_{Q^-}u_{yy}^2\,dxdy+\frac{\gamma_2 p_2}{2}\int\limits_{Q^+}u_{yy}^2\,dxdy\\
&\quad +\sum_{i=1}^{n}\int\limits_{Q^-}u_{x_iy}^2\,dxdy+\sum_{i=1}^{n}\int\limits_{Q^+}u_{x_iy}^2\,dxdy+\varepsilon\sum_{i=1}^{n}\int\limits_{\Omega}u_{x_iy}^2(x,-0)\,dx\\
&\quad =\int\limits_{\Omega}\Big[-\frac{1}{2}\Delta_x a_2(x)u_y^2(x,-0)+\frac{1}{2}\Delta_x b_1(x)u_y^2(x,+0)\Big]\,dx\\
&\quad +\sum_{i=1}^{n}\int\limits_{\Omega}[a_{1x_i}u(x,+0)u_{x_iy}(x,-0)+\gamma_2 b_{2x_i}u_y(x,-0)u_{x_iy}(x,+0)]\,dx\\
&\quad +\int\limits_{Q^-}c(x,y)u_{yy}u\,dxdy-\gamma_2\int\limits_{Q^+}c(x,y)u_{yy}u\,dxdy.
\end{aligned}$$

Applying the Young inequality and using the conditions of the theorem and the integral conditions of the form (13), we arrive at the estimate

$$\begin{aligned}
&\int\limits_{Q^-}u_{yy}^2\,dxdy+\sum_{i=1}^{n}\int\limits_{Q^-}u_{x_iy}^2\,dxdy+\int\limits_{Q^+}u_{yy}^2\,dxdy+\sum_{i=1}^{n}\int\limits_{Q^+}u_{x_iy}^2\,dxdy\\
&\qquad +\varepsilon\sum_{i=1}^{n}\int\limits_{\Omega}u_{x_iy}^2(x,-0)\,dx\le M_5.
\end{aligned}\tag{28}$$

The third a priori estimate

$$\sum_{i,j=1}^{n}\int\limits_{Q^-}u_{x_ix_j}^2\,dxdy+\sum_{i,j=1}^{n}\int\limits_{Q^+}u_{x_ix_j}^2\,dxdy\le M_6\tag{29}$$

is an obvious consequence of the first two estimates and the second main inequality for elliptic operators [7].

The estimates (19), (28), and (29) imply that we can choose convergent subsequences $\{\varepsilon_m\}$ and $\{u_m(x,y)\}$ and there exists a function $u(x,y)$ which is a solution to (1) satisfying (2), (3), and the conjugate conditions of Problem III. The proof of the theorem is complete.

As examples of the operators B_1 and B_2 with the required properties, we can take integral operators taking a function $v(x,y)$ into the functions

$$\begin{aligned}
(B_iv)(x)&=\int\limits_{\Omega}K_{1i}(x,\xi)v(\xi,+0)\,d\xi+\int\limits_{\Omega}K_{2i}(x,\xi)v(\xi,-0)\,d\xi\\
&\quad+\int\limits_{Q^-}N_{1i}(x,\xi,\eta)v(\xi,\eta)\,d\xi d\eta+\int\limits_{Q^+}N_{2i}(x,\xi,\eta)v(\xi,\eta)\,d\xi,
\end{aligned}$$

where $K_{1i}(x,\xi)$, $K_{2i}(x,\xi)$, $N_{1i}(x,\xi,\eta)$, and $N_{2i}(x,\xi,\eta)$, $i = 1, 2$, are given.

The conditions on these kernels sufficient for the fulfillment of the conditions of Theorems 1–6 can be easily found on using the Hölder inequality.

REMARK. The operator Δ_x in the problem under study can be replaced with a general elliptic operator of $x_1, \dots, x_n$ of order $2m$ ($m > 0$ is an integer) complemented with the corresponding collection of boundary conditions on S).

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MODELING THE BOUNDARY EFFECT IN A CYLINDRICAL SHELL UNDER CREEP CONDITIONS

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Abstract. We study the distributions of deformation velocities and stress in neighborhoods of the ends of a shell (the boundary effect) using Rabotnov's two-layer shell model. We solve the system of ordinary differential equations using an iterative procedure and compare solutions for a shell of finite length and a semi-infinite shell.

Keywords: boundary effect, cylindrical shell, creep, iterative procedure

Introduction

Rabotnov proposed in [1] a two-layer shell model to solve elastic-plastic deformation problems. The model is generalized in [2, 3] to the case of deformations of shells under creep conditions. In addition, [3] pointed out a class of shells for which the application of the two-layer model cannot lead to large errors when we solve concrete problems. This class of shells is described by the so-called technical theory of shells which includes the theory of axially symmetric deformations of a circular cylindrical shell, the theory of long cylindrical shells, and the theory of pure bending of curvilinear pipes. Application of the two-layer shell model enabled the development of an efficient numerical method for solving the problems of the stressed and deformed state of shells under creep conditions. In the mechanics of deformable solids, the creep of a material means the property that its deformation may increase with time even under constant stress. Some generalizations of the two-layer model to the case of shells reinforced by edges appeared in [4, 5].

In this article we study the distribution of deformation velocities and stress in neighborhoods of the ends of the shell (the boundary effect). Information on the distributions of deformation velocities and stress in the boundary effect zone enables us to suitably choose test functions while solving problems concerned with critical time with the use of the variational principle [6].

1. Rabotnov's Two-Layer Shell Model

Constructing a simplified theory of shells, we replace the real shell of thickness $2H$ by a two-layer model. This shell consists of two layers of thickness δ each with the distance between their midpoints equal to $2h$ (Fig. 1). Assume that the layers are connected to each other so that the link transmits the shifting force but is unaffected by stretching forces and torques. Therefore, the layers deform in accordance with the Kirchhoff–Love hypothesis. As the reference surface we take a surface lying between the layers. Since the base layers are thin, we can neglect the change in the deformation within the limits of their thickness. We can guarantee that the

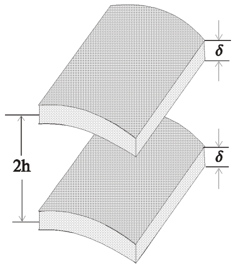

Fig. 1. Two-layer model of shell

condition $\delta \ll h$ holds by choosing appropriate constants in the law of creep for the real shell and the two-layer model [3].

We establish the dependence between the parameters δ and h of the model and the thickness $2H$ of the real shell basing on the requirement that the behavior of the model and that of the real shell coincide in two cases: for a zero-momentum stressed state and for cylindrical bending. Say, for the case of a power law creep with exponent equal to n, these dependences are [2]

$$h = H, \quad \delta = \left(\frac{n}{1+2n}\right)^{\frac{n}{n+1}} H. \tag{1}$$

As n varies from 1 to ∞, the quantity h changes in the interval $1 \leq h \leq H/\sqrt{3}$. The value $n = \infty$ corresponds to the case of ideal plasticity.

In the case of axially symmetric buckling of a circular cylindrical shell under the action of interior pressure and compressing axial force, the problem reduces to the following system of differential equations in dimensionless variables:

$$m'' - u\left(\frac{1}{\omega^+} + \frac{1}{\omega^+}\right) - \frac{2}{\sqrt{3}}\tau + 2p = 0, \tag{2}$$

$$u'' + (m+\tau)\omega^+ - (m+\tau)\omega^- = 0, \tag{3}$$

$$(\omega^\pm)^{\frac{2n}{n-1}} = u^2 + m^2(\omega^\pm)^2, \tag{4}$$

where

$$\omega^\pm = \frac{\sigma_* v^\pm}{\varepsilon_* s^\pm}, \quad u = \frac{\varepsilon_2}{\varepsilon_*}, \quad m = \frac{\sqrt{3}}{4}\frac{M_{11}}{h\delta\sigma_*}, \quad p = -\frac{q_n R}{2\delta\sigma_*}, \quad \tau = \frac{\sqrt{3}}{4}\frac{T_{11}}{\delta\sigma_*},$$

while ε_2 is the velocity of hoop deformation of creep, M_{11} is the longitudinal torque, q_n is the interior pressure, T_{11} is the longitudinal compressing force, v^+, s^+ and v^-, s^- are the intensivities of the creep velocity and stress in the upper and lower base layers respectively, ε_* and σ_* are characteristic quantities with the dimensions of creep velocity and stress respectively; the primes indicate differentiation with respect to the dimensionless longitudinal coordinate $\xi = x/b$, with $b^2 = 4/(\sqrt{3}Rh)$, and R is the radius of the cylindrical shell.

Equation (4) determines $\omega^\pm$ as functions of u and m.

When u and m vary independently, (2)–(4) are the Euler equations for the functional

$$N = \int_0^l \left[u'm' + \frac{1}{2}\psi(\omega^+) + \frac{1}{2}\psi(\omega^-) - (m+\tau)^2\omega^+ - (m-\tau)^2\omega^- + 2\left(\frac{\tau}{\sqrt{3}} - p\right)u\right] d\xi, \tag{5}$$

where

$$\psi(\omega) = \frac{1}{\varepsilon_*^2} \int \frac{d(v^2)}{\omega}$$

and $l = L/b$ is the dimensionless length of the shell; for an elastic shell, b is the size of the boundary effect zone.

2. An Iterative Procedure for Solving the Boundary Value Problem on the Creep Deformation of a Cylindrical Shell

We must supplement (2)–(4) with boundary conditions whose form depends on the method of fixing the ends of the shell. If both ends are rigidly restrained then the boundary conditions are

$$u(0) = u'(0) = u(l) = u'(l) = 0. \tag{6}$$

If both ends are hinged then the boundary conditions

$$u(0) = m(0) = u(l) = m(l) = 0 \tag{7}$$

express the vanishing of displacement and momentum. Other kinds of boundary conditions are possible.

Let us also write down boundary conditions for a semi-infinite shell. On the left edge of the shell we can impose conditions (6) or (7). At infinity the shell is in the zero-momentum state. Consequently, $m(\infty) = 0$ and $u(\infty) = u_\infty$. Put $\omega^+(\infty) = \omega^-(\infty) = \omega_\infty$. By (4),

$$\frac{2u_\infty}{\omega_\infty} = 2\left(p - \frac{\tau}{\sqrt{3}}\right), \quad (\omega_\infty)^{2n/(n-1)} = u_\infty^2 + (\tau\omega_\infty)^2.$$

Consequently,

$$u_\infty = \left(p - \frac{\tau}{\sqrt{3}}\right)\omega_\infty = \left[\left(p - \frac{\tau}{\sqrt{3}}\right) + \tau^2\right]^{2n/(n-1)}. \tag{8}$$

Rearrange (2) and (3) as

$$m'' - 2\frac{u}{\omega_\infty} = \frac{\omega_\infty - \omega^+}{\omega^+\omega_\infty}u + \frac{\omega_\infty - \omega^-}{\omega^-\omega_\infty}u + 2\left(\frac{\tau}{\sqrt{3}} - p\right), \tag{9}$$

$$u'' + 2m\omega_\infty = m(\omega_\infty - \omega^+) + m(\omega_\infty - \omega^-) - \tau(\omega^+ - \omega^-). \tag{10}$$

Assuming that the right-hand sides of (9) and (10) are known functions of ξ and integrating the system subject to the appropriate boundary conditions, we obtain

$$\begin{aligned} u = {} & [A - J_1(\xi)]\delta(\xi) + [J_2(\xi) - B]\gamma(\xi) + [C - J_3(\xi)]\beta(\xi) \\ & +[D + J_4(\xi)]\alpha(\xi) - \frac{\tau}{\sqrt{3}}\omega_\infty + p\omega_\infty, \end{aligned} \tag{11}$$

$$\begin{aligned} m = {} & \frac{1}{\omega_\infty}\{[[A - J_1(\xi)]\gamma(\xi) - [J_2(\xi) - B]\delta(\xi) \\ & -[C - J_3(\xi)]\alpha(\xi) + [D + J_4(\xi)]\beta(\xi)]\}, \end{aligned} \tag{12}$$

where

$$J_1(\xi) = \int\limits_0^\xi [F^+\alpha(\xi) + F^-\beta(\xi)]\,d\xi, \quad J_2(\xi) = \int\limits_0^\xi [F^+\beta(\xi) - F^-\alpha(\xi)]\,d\xi,$$

$$J_3(\xi) = \int\limits_0^\xi [F^+\gamma(\xi) - F^-\beta(\xi)]\,d\xi, \quad J_4(\xi) = \int\limits_0^\xi [F^+\delta(\xi) + F^-\gamma(\xi)]\,d\xi,$$

$$F^\pm = \frac{1}{4}\left\{\left[\frac{\omega_\infty - \omega^+}{\omega^+} + \frac{\omega_\infty - \omega^-}{\omega^-}\right]u \pm (\omega^+ - \omega_\infty)(m+\tau) \pm (\omega^- - \omega_\infty)(m-\tau)\right\},$$

$$\alpha(\xi) = \exp(-\xi)\sin\xi,\ \beta(\xi) = \exp(-\xi)\cos\xi,\ \gamma(\xi) = \exp(\xi)\sin\xi,\ \delta(\xi) = \exp(\xi)\cos\xi.$$

The constants A, B, C, and D depend on the form of boundary conditions. For (6) the constants are defined as

$$B = -D, \quad A = -r - c, \quad C = \frac{(\beta(\xi) - \delta(\xi))\Omega_1(\xi) - (\alpha(\xi) - \gamma(\xi))\Omega_2(\xi)}{\delta(\xi)},$$

$$D = \frac{(\beta(\xi) - \delta(\xi))\Omega_2(\xi) + (\alpha(\xi) + \gamma(\xi))\Omega_1(\xi)}{\delta(\xi)},$$

$$\delta(\xi) = (\beta(\xi) - \delta(\xi))^2 + (\alpha(\xi) + \gamma(\xi))^2,$$

$$\Omega_1(\xi) = r(\delta - 1) + J_1(l)\delta - J_2(l)\gamma + J_3(l)\beta - J_4(l)\alpha,$$

$$\Omega_2(\xi) = r\gamma + J_1(l)\gamma + J_2(l)\delta - J_3(l)\alpha - J_4(l)\beta, \quad r = \left(p - \frac{\tau}{\sqrt{3}}\right)\omega_\infty.$$

For a semi-infinite shell with hinged left end the constants are

$$A = J_1(\infty), \quad B = J_2(\infty), \quad C = -J_1(\infty) - r, \quad D = -J_2(\infty).$$

For a semi-infinite shell the integrals $J_1(\infty)$ and $J_2(\infty)$ make sense provided that $|u - u_\infty| \to 0$ and $|m| \to 0$ not slower than $\exp(-\xi)$ as $\xi \to \infty$.

We solve (11) and (12) using iterations. As the initial approximation to the functions u and m we take the expressions resulting from (11) and (12) when we put $\omega^\pm = \omega_\infty$, $u = u_\infty$, and $m = 0$ in the right-hand sides. Then we find from (4) the initial approximation to $\omega^\pm$. To find a root of (4), we use Newton's method. Inserting the initial approximation to u and m into the right-hand side of (11) and (12), we find the first approximation. The process then repeats.

3. The Boundary Effect in a Semi-Infinite Cylindrical Shell Loaded by Interior Pressure

Assume that a circular cylindrical shell is loaded by the interior pressure p and $\tau = 0$. Assume that the edges of the shell for $\xi = 0$ are clamped. Since $\tau = 0$, it follows from (4) that

$$\omega^+ = \omega^- = \omega \tag{13}$$

and the function ω is determined from

$$\omega^{2n/(n-1)} = u^2 + m^2\omega^2. \tag{14}$$

Then (9) and (10) become

$$m'' - 2\frac{u}{\omega_\infty} = 2\frac{\omega_\infty - \omega}{\omega\omega_\infty}u - 2p, \tag{15}$$

$$u'' + 2m\omega_\infty = 2m(\omega_\infty - \omega), \tag{16}$$

where $\omega_\infty = p^{n-1}$.

Changing the variables

$$u = p^n \bar{u}, \quad m = p\overline{m}, \quad \omega = p^{n-1}\overline{\omega},$$

we rearrange (14)–(16) as

$$\overline{\omega}^{2n/(n-1)} = \bar{u}^2 + \overline{m}^2\overline{\omega}^2, \tag{18}$$

$$\overline{m}'' - 2\frac{\bar{u}}{\overline{\omega}_\infty} = 2\frac{\overline{\omega}_\infty - \overline{\omega}}{\overline{\omega}\overline{\omega}_\infty}\bar{u} - 2, \tag{19}$$

$$\bar{u}'' + 2\overline{m}\overline{\omega}_\infty = 2\overline{m}(\overline{\omega}_\infty - \overline{\omega}). \tag{20}$$

In the new variables $\overline{\omega}_\infty = \bar{u}_\infty = 1$, and the only remaining parameter of the problem is the creep exponent n.

Replace (19) and (20) with integral equations (11) and (12), which we should simplify according to (13). Also, replace $p\omega_\infty$ with 1. We have

$$F^\pm = \frac{\overline{\omega}_\infty - \overline{\omega}}{2}\left(\frac{\bar{u}}{\overline{\omega}} \pm m\right).$$

For the initial approximation we obtain the expressions

$$\bar{u}_0 = 1 - \exp(-\xi)(\cos\xi - \sin\xi), \quad \overline{m}_0 = -\exp(-\xi)(\cos\xi - \sin\xi).$$

To determine $\overline{\omega}$, we use Newton's method:

$$\overline{\omega}_{k+1} = \overline{\omega}_k + \frac{(1/4)\bar{u}_k^2 + \overline{m}_k^2\overline{\omega}_k^2 - \overline{\omega}_k^{2n/(n-1)}}{(2n/(n-1))\overline{\omega}_k^{(n+1)/(n-1)} - 2\overline{m}_k^2\overline{\omega}_k^2}.$$

We use these formulas to determine $\overline{\omega}$ at the points of the interval under consideration from the prescribed values of $\bar{u}$ and $\overline{m}$. For $\xi = 0$ we can find from (14) the value of $\overline{\omega}$ in terms of $\overline{m}$ as

$$\overline{\omega}(0) = (\overline{m}(0))^{n-1}.$$

At the subsequent points of the interval we take as the initial approximation to $\overline{\omega}$ the solution to (14) at the previous point. The process of successive approximations continues until the inequality $|\overline{\omega}_{k+1} - \overline{\omega}_k| \leq 0.00001$ becomes valid. Since the initial approximation at every point differs little from the exact value, three to four approximations turn out sufficient to satisfy the last inequalities. To calculate the integrals $J_1(\infty)$ and $J_2(\infty)$, we replace the semi-infinite interval by $\xi \in [0, 7]$. As a consequence, the values of $\bar{u}$, $\overline{m}$, and $\overline{\omega}$ differ from their values at infinity by at most 0.0001. We divide the interval of integration into intervals with $\Delta\xi = 0.05$ and use the trapezoid formula. The difference between the values of the required functions at the fourth and fifth nodes of approximation amounts to the third digit after the decimal point, between the values at the sixth and seventh nodes, to the fifth digit, and between those on the eleventh and twelfth nodes, to the eighth digit.

4. The Results of Modeling

Fig. 2 depicts the dependence of the dimensionless momentum $\overline{m}$ and dimensionless velocity $\bar{u}$ of hoop deformation on the coordinate ξ for the creep exponents $n = 1, 3, 5$. The graphs imply that as n increases, the boundary effect zone somewhat grows, and simultaneously we observe a smoother change of momentum in the boundary effect zone. Our computations show that the momentum $\overline{m}$ in clamped edges unchanged as the creep exponent n varies, and in fact equals 1. The same result is obtained in [3] with the use of the variational principle (the functional in (5)). This fact is natural; for instance, if we use Mises' fluidity criterion then in the limit state of the shell as $n = \infty$ the momentum also equals 1. Indeed, for $\xi = 0$ the velocity of hoop deformation vanishes. Thus, $\sigma_2 = (1/2)\sigma_1$, and Mises' criterion

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = s_\infty^2$$

implies that $\sigma_1 = (2/\sqrt{3})s_\infty$. Therefore, $M_1(0) = (4/\sqrt{3})\delta h s_\infty$ or $\overline{m} = 1$ since the limit state is reached for $p = 1$.

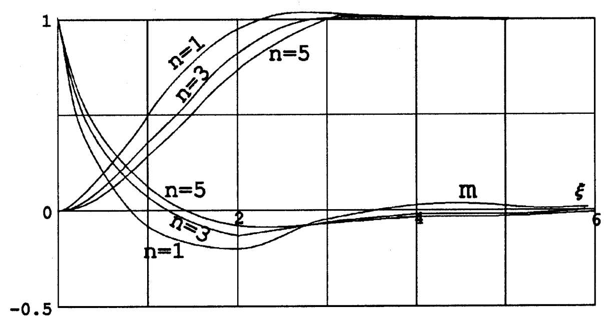

Fig. 2. Distribution of deflection and momentum in semi-infinite cylindrical shell

For the original shell of thickness H we express torque as

$$\frac{3M_1}{2qHa} = -\left(\frac{n}{1+2n}\right)^{n/(n+1)}\sqrt{3}\overline{m}.$$

This follows once we use the definition of $\overline{m}$ and (1), which yield the dependence of the parameters of the model on the thickness of modeled and the creep exponent.

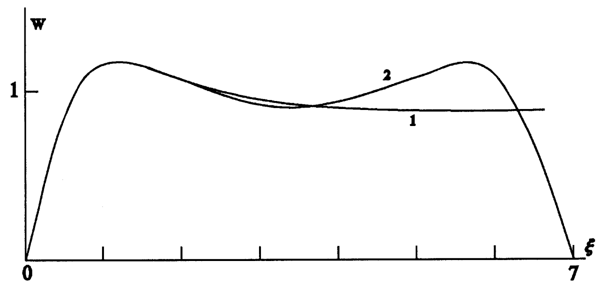

Fig. 3. Distribution of deflection in finite and infinite cylindrical shells

Fig. 3 illustrates a comparison of the dependence of the transverse velocity of deformation on the coordinate ξ for finite and infinite shells.

We can express the longitudinal and hoop stresses in terms of $\overline{m}$ and $\bar{u}$ as follows [3]:

$$\frac{1}{p}\frac{\sigma_1^\pm}{\sigma_*} = \pm\frac{2}{\sqrt{3}}\overline{m}, \quad \frac{1}{p}\frac{\sigma_2^\pm}{\sigma_*} = \frac{\bar{u}}{\bar{\omega}} \pm \frac{\overline{m}}{\sqrt{3}}.$$

This implies that p is the value of hoop stress at infinity, $(\sigma_2(\infty)/\sigma_*) = p$.

Fig. 4 depicts the dependence of hoop stresses on the dimensionless coordinate ξ. The dashed lines correspond to the stress $(1/p)(\sigma_2^+/\sigma_*)$, while the solid lines, to the stress $(1/p)(\sigma_2^-)/\sigma_*)$ in the lower layer.

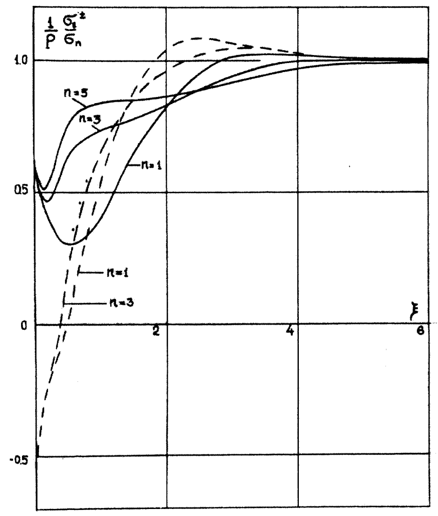

Fig. 4. Distribution of stresses in the boundary effect zone

It is particularly interesting to estimate the maximal stretching stress since large stretching stresses can lead to cracks in a construction. The maximal stretching stress is a longitudinal stress on the interior side of the shell. The maximal stretching stress in a modeled shell is different from the maximal stress in the model, and we can find it in terms of torque at the restraint:

$$(\sigma_1)_{\max} = \frac{M_1(0)}{W_n}, \quad W_n = \frac{I_n}{h^\nu}, \quad I_n = \frac{h}{1+1/\nu}, \quad \nu = \frac{1}{n}.$$

Refer as the *concentration coefficient* to the ratio of the maximal stretching stress to the hoop stress at infinity. The last equalities yield

$$\frac{(\sigma_1)_{\max}}{\sigma_2(\infty)} = (\nu+2)^{\nu/(1+\nu)} \frac{2}{\sqrt{3}} \overline{m}(0).$$

For $n = 1$ the maximal stretching stress at the restraint exceeds the hoop stretching stress at infinity by a factor of two. The results of computation imply that the concentration coefficient decreases with the growth of the creep exponent. But its value remains quite large for the creep exponent for materials widely used in technology. As a consequence, it is necessary to take the boundary effect into account when calculating constructions.

Conclusion. Using the two-layer shell model we model the distribution of stresses and velocity of deformation in the boundary effect zone in cylindrical shells, both of finite length and semi-infinite.

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VARIOUS APPROACHES TO MODELING INDUCED CURRENTS IN TRANSMISSION LINES

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Abstract. We compare the currents and voltages induced in an aerial transmission line in the event of a nearby lightning strike, calculating them with the use of two different mathematical models. The first model describes the electrostatic component of the induced currents, and the second, the electromagnetic component. We show that the peak values of these components are comparable and that in permafrost conditions the peak values of the electrostatic components of induced currents and voltage can be greater by orders of magnitude, and so more dangerous than in the areas without permafrost.

Keywords: lightning, transmission line, permafrost, induced voltage, induced currents

The problem of electromagnetic compatibility of technical constructions with atmospheric electricity is particularly urgent in permafrost conditions due to the low conductivity of the ground. Some aspect of this problem concerns the questions of improving the effectiveness of the electrical protection of long-distance transmission lines. These include pipelines, power transmission lines, and communication lines. From the physical viewpoint, the long-distance lines amount to long conductive lines in a stratified medium (air, conducting ground, or permafrost) and subject to electric induction during thunderstorms and intense geomagnetic disturbances. In long-distance transmission lines, the currents and voltages appear both in the absence and in presence of a direct strike lightning. These are called induced currents and voltages. There are two kinds of them: Those of the first kind are electromagnetic, i.e., they arise in result of the influence of the electromagnetic field of exterior disturbances. Those of the second kind are electrostatic: the electrostatic field of a thundercloud induces electric charges on long-distance conducting lines. The fast discharge of a thundercloud releases these charges; flowing along the transmission line, and they form a current and voltage wave. The use of long-distance power transmission lines and communication lines in Yakutia during the last decades shows that induced currents and voltages cause many emergencies. Some operational parameters of the electric protection of these lines often fail to reach the required standardized values.

In order to draw up recommendations on the protection of transmission lines against the action of atmospheric electricity, it is necessary first of all to analyze the processes, in particular, to estimate the induced currents and voltages. The standard methods for approximate calculation of induced overvoltages in aerial lines rest on calculating the electromagnetic field of the lightning channel. In essentially all published articles, the calculation of induced currents and voltages accounts only

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for electromagnetic induction and neglects the electrostatic part. But, in our opinion, in domains with high specific electric resistance of the ground, including domains with permafrost, it is necessary to account for the electrostatic component (i.e., the current and voltage wave) while calculating the induced currents and voltages. The authors are aware of few articles calculating current and voltage waves.

Let us survey some basic facts of use below. The electrization of thunderclouds in the most cases (up to 90%) results in lightning carrying negative charge to the ground. In the middle latitudes, about 30–40% of the total number of lightnings strike the ground. The remaining 60–70% constitute discharges between clouds or their oppositely charged parts. The lightning consists mostly of several separate successive discharges (up to 14). A multicomponent lightning can last up to 1 sec. Most often the duration of a lightning strike is below 0.1 sec. The failures in long-distance lines occur mainly due to a nearby lightning strike.

The articles [1–7] consider some mathematical models of the electromagnetic influence of lightning discharges to the ground or between clouds on a long-distance line. Tsapenko attempted [8] to construct a mathematical model of current and voltage waves in power transmission lines; To model the initial potential of a cloud, he used a quadratic function without justification. The most adequate models of current and voltage waves in a coaxial cable under permafrost conditions appeared in [9–12]. We constructed and solved analytically and numerically the mathematical models of current and voltage waves induced in power transmission lines by lightning discharges in permafrost conditions [13–20].

In this article we compare the calculations of currents and voltages induced in a power transmission line according to our model [13–20] to the model described in [3].

Let us briefly describe our model of current and voltage waves. We construct it on assuming the instantaneous character of the lightning discharge. For simplicity, consider a transmission line with one infinitely thin conductor (Fig. 1) characterized by the following distributed parameters: resistance R, capacity C, inductance L, and leakage factor G calculated per unit length.

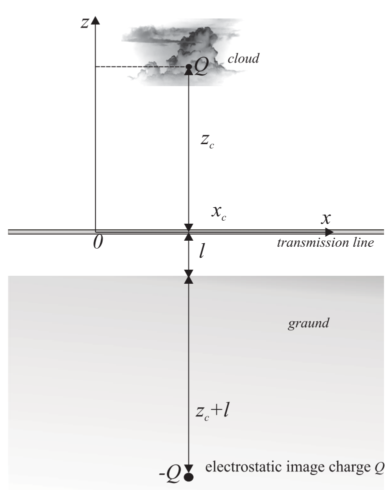

Fig. 1. Lightning strike near transmission line

Introduce the system of Cartesian coordinates x, y, and z. Suppose that the domain $z < l$ is occupied by conducting ground, where the parameter l characterizes the thickness of the permafrost layer. The transmission line is aligned with the x-axis. We model a thundercloud with a point charge Q with coordinates (x_c, y_c, z_c). Consequently, the line lies in the field of the charge Q and its electrostatic image $-Q$. Assume that the conductor is grounded at an infinitely distant point, while the potential v of the ground equals 0, i.e., the charges induced on the line create the potential compensating the potential of two point charges Q and $-Q$.

At time $t = 0$ the charges Q disappear instantaneously (the cloud discharges), and the current wave, the flow of induced charges, propagates along the line for $t > 0$. It is described by the telegraph system of equations. Thus, to determine the current $i(x,t)$ and the voltage $u(x,t)$ in the lines following the discharge of a thundercloud, we obtain the problem:

$$\begin{cases} u_x + Li_t + Ri = 0, \\ i_t + Cu_t + Gu = 0, \\ i(x,0) = 0, \\ u(x,0) = f(x), \end{cases} \quad -\infty < x < \infty,\ t > 0. \tag{1}$$

We find the initial function $f(x)$ from the previous arguments:

$$f(x) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{\sqrt{z_c^2 + (x - x_c)^2 + y_c^2}} - \frac{Q}{\sqrt{(2l + z_c)^2 + (x - x_c)^2 + y_c^2}}\right),$$

this is the potential that up to time $t = 0$ compensates the potential of the charges Q and $-Q$. The second initial condition $i(x,0) = 0$ is obvious.

The solution to our problem is the voltage

$$u(x,t) = \frac{1}{2}e^{\lambda t}(f(x + at) + f(x - at))$$
$$+\frac{1}{2}e^{\lambda t}\int\limits_{x-at}^{x+at}\left(\frac{\mu}{a}I_0(k\sqrt{a^2t^2 - (x-y)^2}) + kta\frac{I_1(k\sqrt{a^2t^2 - (x-y)^2})}{\sqrt{a^2t^2 + (x-y)^2}}\right)f(y)\,dy \tag{2}$$

and the current

$$i(x,t) = -\sqrt{\frac{C}{L}}\frac{e^{\lambda t}}{2}\int\limits_{x-at}^{x+at} I_0(k\sqrt{a^2t^2 - (x-y)^2})f_y(y)\,dy, \tag{3}$$

where $I_0(z)$ and $I_1(z)$ are modified Bessel functions.

If the condition

$$RC = LG \tag{4}$$

of the absence of disturbances is fulfilled then the solution to our problem simplifies, becoming

$$u(x,t) = \frac{1}{2}e^{-\frac{R}{L}t}(f(x - at) + f(x + at)), \tag{5}$$

$$i(x,t) = \frac{1}{2}\sqrt{\frac{C}{L}}e^{-\frac{R}{L}t}(f(x - at) - f(x + at)). \tag{6}$$

Let us recall how the numerical experimentation of [3] studied the influence of low conductivity of the ground on the induced currents in a power transmission line, accounting only for the electromagnetic component of induced currents. It all starts with calculation of the electromagnetic field of the lightning channel by several methods. We determine the horizontal component of the electric field near a power transmission line without imposing boundary conditions on the wires. Then we insert the strength of this field into the right-hand side of the telegraph equations as the external electromotive force and solve the resulting initial-boundary value problem. The power transmission line was treated in [3] as one wire of radius $b = 0.005$ m hanging at $l = 10$ m above ground. Furthermore, $C = \frac{2\pi\varepsilon_0}{\log(\frac{2l}{b})} = 6.7 \cdot 10^{-12}$ F/m is the capacity, $R = \rho\frac{l}{\pi b^2} = 3.4 \cdot 10^{-4}$ Ohm/m is the resistance, and $L = \frac{\mu_0}{2\pi}\log\left(\frac{2l}{b}\right) = 1.6 \cdot 10^{-6}$ H/m is the inductance of the conductor per unit length.

We calculate the leakage factor G as

$$G = \frac{\sigma_0}{\varepsilon_0}C,$$

where σ_0 is the conductivity of air. We can justify this formula in the electrodynamic derivation of telegraph equations [21].

The vertical lightning strikes near a power transmission line at distance $r = 50$ m from its center (Fig 2).

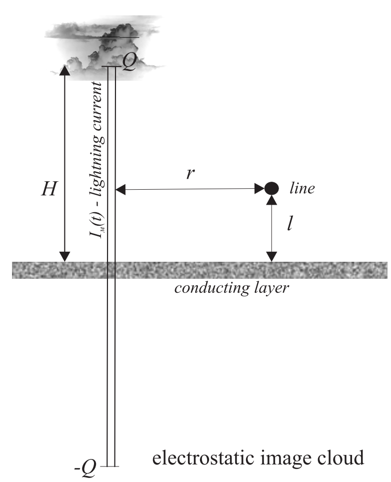

Fig. 2. The lightning geometry calculated in [13]

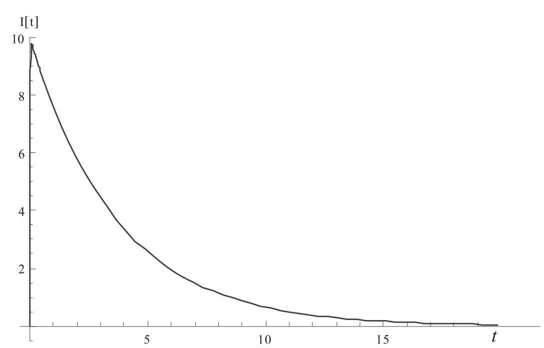

Fig. 3. Lightning current pulse

To calculate the horizontal component of the electric field near a power transmission line, assume that we express the lightning current at the base of the lightning channel as

$$I(t) = I_0(e^{-\alpha t} - e^{-\beta t}), \tag{7}$$

where $I_0 = 12$ kA, $\alpha = 3 \cdot 10^4\ s^{-1}$, and $\beta = 10^7\ s^{-1}$.

Fig. 3 depicts the current pulse. We assume that the velocity of propagating lightning current equals $v = 1.3 \cdot 10^8\ m/s$.

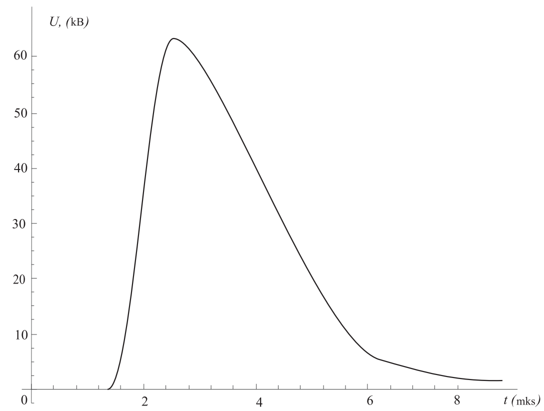

Fig. 4. Voltage in a power transmission line induced by the electromagnetic field of lightning current

The results of [3] show that the peak values of voltage in a power transmission line can reach 60 kV. Fig. 4 depicts the pulse of induced overvoltage in the power transmission line.

Let us now find the predicted parameters for our model current and voltage wave with instantaneous lightning discharge using the original data of [3]. The total charge carried by the lightning current (7) for our model will be the charge

$$Q = \int\limits_0^\infty I(t)\, dt = 0.33\ C$$

of the cloud. For a lightning current pulse lasting 11.5 μs and the specified velocity of the current, we estimate the height of the lightning as about $H = 1500$ m. Therefore, we find the expression

$$f(x) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{\sqrt{(H-l)^2 + x^2}} - \frac{Q}{\sqrt{(H+l)^2 + x^2}}\right)$$

for the initial potential of our model. Fig. 5 depicts the results of calculations of the current and voltage wave for a long power transmission line with the above parameter when lightning strikes the ground 50 m away from all power transmission lines.

Calculations show that the peak value 60 kV for current and voltage waves agrees with the results of [3] on the order of magnitude.

The authors previously constructed some mathematical model of current and voltage wave in a transmission line in permafrost conditions [13–20]. Adding to this problem a permafrost layer with $l_p = 250$ m, we obtain the predicted results,

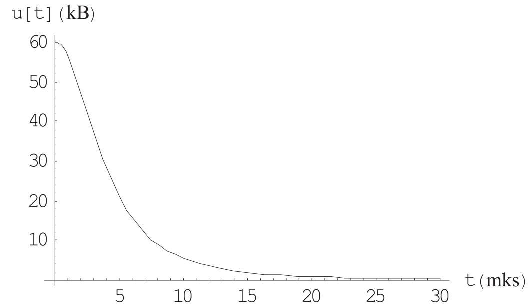

Fig. 5. Voltage wave in a power transmission line according to our model with instantaneous lightning discharge

presented in Fig. 6. The initial potential, taking permafrost into account, is of the form

$$f(x) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{\sqrt{(H-l_p)^2+x^2}} - \frac{Q}{\sqrt{(H+l_p)^2+x^2}}\right).$$

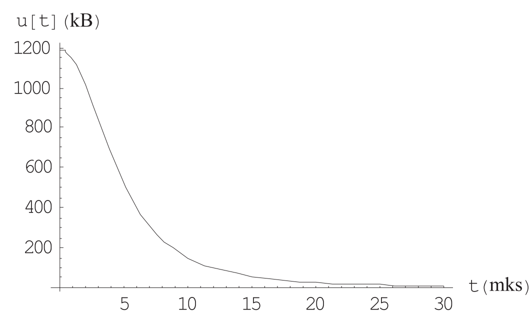

Fig. 6. Voltage wave in a power transmission line in permafrost conditions taking into account the instantaneous character of lightning strike

It is clear that in permafrost conditions the induced voltage can be dozens of times more dangerous.

Conclusions

• The peak values of the electrostatic component of induced voltage (current and voltage wave) are comparable to those of the electromagnetic components.

• In permafrost conditions the peak values of the electrostatic component of induced voltage (current and voltage wave) can be greater by orders of magnitude, and so more dangerous than in the areas without permafrost.

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MATHEMATICAL MODELING OF JIGGING

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Abstract. We present a mathematical model of jigging using the statical approach for describing the process and the theory of Brownian motion. The Fokker–Planck equation is obtained for fractions in a jigging machine. The distributions of the grainy rocks under study are calculated in various cases.

Keywords: jigging, distribution, Fokker–Planck equation, diffusion, probability, model

Jigging is a method of gravity-assisted enrichment of ores which is based on stratifying grainy rock fractions by density and size. It goes in a vibrating medium under the action of the gravitational field of the Earth. In this article we discuss the developing of a mathematical model for jigging.

Statistical approach is often used to describe the jigging process: we consider not only deterministic mechanical processes but also stochastic. In [1–4] it is not shown adequately which premises and assumptions lead to the Fokker–Planck equation, and how we solve the latter. In this article we suggest a physical model of jigging and, basing on it, obtain a relation of Fokker–Planck type. To solve this equation, we use a model of the motion of a Brownian particle in the field of the Earth [5]. The solution extends to the case of a set of noninteracting particles. The equation has unknown parameters to be determined from the physical model and refined as a result of physical modeling.

Consider the following problem:

We put N identical balls of density ρ_1 and radius r_0 into a jigging bed, submerging them into a medium of density ρ_v and viscosity η_v. At a certain height h there is a ball of the same size but a greater density $\rho_0 > \rho_1$ (Fig. 1). As the jigging bed oscillates with frequency ω_0 and amplitude a_0, the test particle must diffuse downward due to the action of gravity. We need to find the probability of the particle occurring at a certain location at arbitrary time.

Assume that the problem is one-dimensional.

Firstly, consider the case that gravity is absent, together with the boundaries; i.e., the system is unbounded. At the initial moment the particle is at rest; when the system starts oscillating, the particle will move according to Markov processes. A Brownian particle is moving chaotically under the action of the thermal motion of the molecules of the surrounding medium. Furthermore, the molecules bombarding it have certain average velocity in accordance with the temperature. In our case, the role of these molecules belongs to the density ρ_1 balls surrounding our test ball and moving under the action of exterior periodic forces. The balls of the surrounding medium have kinetic energy in accordance with the oscillations of the medium, namely, $\frac{m\langle v\rangle^2}{2} = \frac{ma_0^2\omega_0^2}{2}$, where a_0 and ω_0 are the amplitude and frequency of oscillations of the medium. The two kinds of forces act on the particle: the gradient

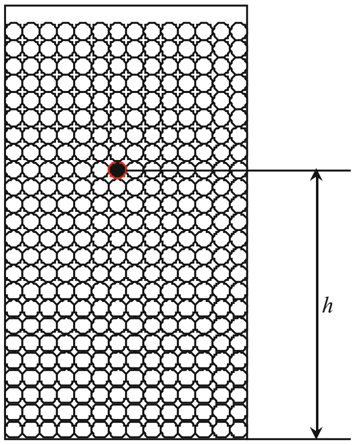

Fig. 1

forces F_{gr}, since we treat each moment as a statistical ensemble and the resistance forces. The latter are caused by collisions, F_1, of the particle under consideration with the surrounding balls and viscosity F_2 when the process occurs in water, for instance. The resistance force F_s equals $F_1 + F_2$.

We can determine the resistance force due to collisions as follows: The mean decelerating force equals the loss of momentum. For each collision it completely loses the velocity. This force equals approximately

$$F_1 = -\frac{dp}{dt} = -m\frac{dv}{dt} \approx -m\frac{\bar{v}}{\tau}, \tag{1}$$

where $\bar{v}$ is the mean velocity of particles and τ is the mean free time between collisions.

We can find the mean value of free time from the oscillatory characteristics of the system. The mean squared velocity of particles in the oscillations of the jigging machine equals

$$\langle v \rangle = a_0 \cdot \omega_0. \tag{2}$$

The mean length of free path equals the amplitude of oscillations. Hence, the mean free time equals the period of oscillations

$$\tau = \frac{2\pi}{\omega_0}. \tag{3}$$

Thus,

$$\vec{F_1} \approx -\frac{m\omega_0}{2\pi}\vec{v}. \tag{4}$$

Assume that the resistance of the medium equals the Stokes force

$$\vec{F_2} = -6\pi\eta r_0 \vec{v}, \tag{5}$$

where η is the viscosity of the medium.

To determine the gradient force, consider an inhomogeneous one-dimensional system, that is, a one-dimensional problem, where $n(x)$ depends only on one variable x, a reservoir with inhomogeneous distribution of particles (Fig. 2). Suppose

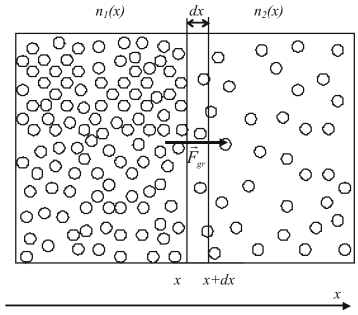

Fig. 2

that the concentration of particles is greater to the left than to the right of some boundary: $n_1 > n_2$.

The force proportional to concentration and equal to $kn_1\Delta S$ acts on the left boundary of the domain dx and equal to $kn_2\Delta S$, on the right boundary, where $k = \frac{m\langle v\rangle^2}{3} = \frac{ma_0^2\omega_0^2}{3}$. The force, acting on a particle inside the domain, equals

$$F_{gr} = \frac{k(n_2 - n_1)\Delta S}{n(t,x)\Delta V} = \frac{k(n(t,x) - n(t,x+dx))\Delta S}{n(t,x)\Delta S dx} = -\frac{k}{n(t,x)}\frac{dn(t,x)}{dx} \tag{6}$$

or

$$\vec{F}_{gr} = -\frac{k}{n(t,\vec{r})}\nabla n(t,\vec{r}) \tag{7}$$

in vector notation.

The mathematical expression for the conservation of amount of substance is the continuity equation

$$\frac{\partial n(t,\vec{r})}{\partial t} + \nabla\cdot(n(t,\vec{r})\cdot\vec{v}) = 0. \tag{8}$$

The balance of the forces

$$\sum_i \vec{F_i} = \vec{F_{gr}} + \vec{F_1} + \vec{F_2} = 0 \tag{9}$$

acting on a given particle yields

$$-\frac{k}{n(t,\vec{r})}\nabla n(t,\vec{r}) - \frac{m\omega_0}{2\pi}\vec{v} - 6\pi\eta r_0\vec{v} = -\frac{k}{n(t,\vec{r})}\nabla n(t,\vec{r}) - \alpha\vec{v} = 0, \tag{10}$$

where $\alpha = \frac{m\omega_0}{2\pi} + 6\pi\eta r_0$.

By (8) and (10),

$$\frac{\partial n(t,\vec{r})}{\partial t} = \frac{k}{\alpha}\Delta n(t,\vec{r}) = D\Delta n(t,\vec{r}), \tag{11}$$

where $D = \frac{k}{\alpha}$ is the coefficient of macrodiffusion.

The resulting expression is the Fokker–Planck equation for a free particle. The solutions to this classical equation are known; in particular, in the one-dimensional case the fundamental solution is

$$n(t,x) = \frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right). \tag{12}$$

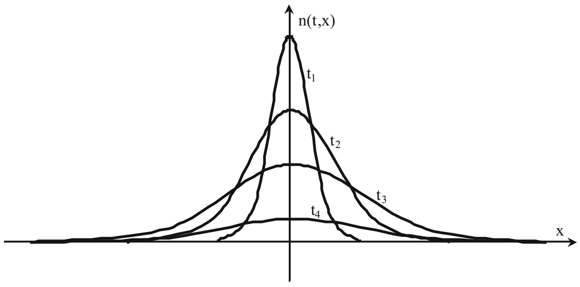

Fig. 3

Fig. 3 shows this dependence for various times.

Returning to the problem under study, consider the motion of a particle in the gravity field of the Earth. The particle is in a reservoir, bounded below by an impermeable wall and surrounded by other particles of smaller density. During oscillations the particle diffuses downward. It is acted on by the gravity force

$$\vec{F_3} = m\vec{g}, \tag{13}$$

where $m = \frac{4}{3}\pi r_0^3 \rho_0$ is the mass of the particle under study. Then the Fokker–Planck equation is of the form

$$\frac{\partial n(t,x)}{\partial t} - \frac{mg}{\alpha}\frac{\partial n(t,x)}{\partial x} - D\Delta n(t,x) = 0 \tag{14}$$

with the initial and boundary conditions

$$n(0,x) = \delta(x-h), \tag{15}$$

$$\int\limits_0^\infty n(t,x)\,dx = 1, \tag{16}$$

$$n(t,x)|_{x\to+\infty} = 0, \tag{17}$$

$$\left.\frac{\partial n(t,x)}{\partial x}\right|_{x\to+\infty} = 0, \tag{18}$$

$$\left(\frac{mg}{k}n(t,x) + \frac{\partial n(t,x)}{\partial x}\right)_{x=0} = 0. \tag{19}$$

The last relation is the condition of the absence of flow through the lower surface.

To solve this problem, using the solution to a similar problem for a Brownian particle [5], we obtain the analytical expression

$$\begin{aligned} n(t,x) = \frac{1}{\sqrt{4\pi Dt}}\left(\exp(-\frac{(x-h)^2}{4Dt}\right) + \exp\left(-\frac{(x+h)^2}{4Dt}\right)\Bigg) \\ \times\exp\left(-\frac{(mg)^2t}{4k\alpha} - \frac{mg(x-h)}{2k}\right) \\ +\frac{mg}{k}\exp\left(-\frac{mgx}{k}\right)\frac{1}{\sqrt{4\pi Dt}}\int\limits_{-\infty}^{\frac{mgt}{\alpha}-x-h}\exp\left(-\frac{\eta^2}{4Dt}\right)d\eta. \end{aligned} \tag{20}$$

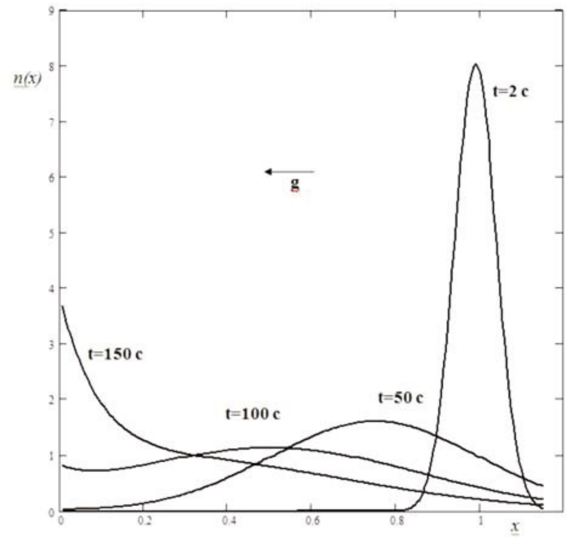

Fig. 4

Fig. 4 depicts the dependence according to (20) at various times. The marks on the horizontal axis indicate the height of the reservoir with balls.

It is clear from Fig. 4 that the distribution of the probability of finding the particle spreads initially while moving down, and eventually becomes the Boltzmann distribution.

This problem is equivalent to the problem with a set of N particles of density ρ_0 distributed on the same plane at some height h (Fig. 5). Then the distribution of the number of particles is described by the curve (20); furthermore, the number of particles is proportional to the area of the figure inside the domain adjacent to the horizontal axis and the line in Fig. 4.

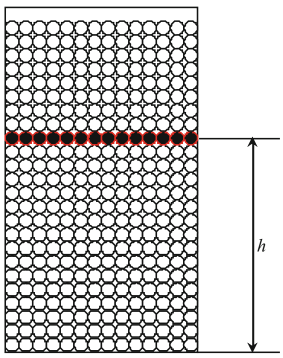

Fig. 5

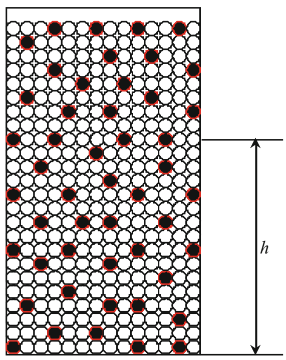

Fig. 6

In reality, the extracted gold pieces lie randomly in the volume of the jigging bed. To model this distribution, we can use a generator of random numbers. When we have sufficiently many grains of the material under study, the initial distribution is roughly uniform. Thus, we consider a uniformly distributed system (Fig. 6).

Fig. 7 shows the dependences for the uniform distribution.

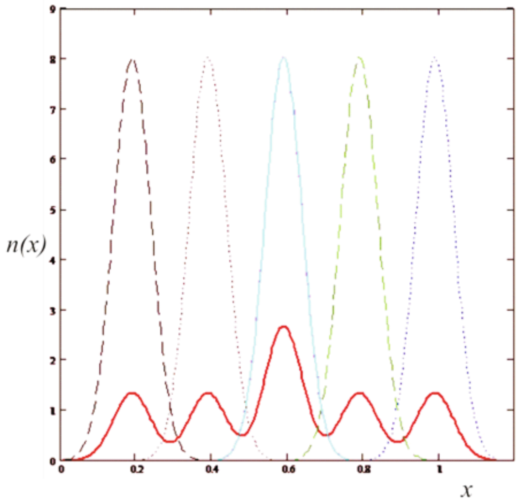

Fig. 7

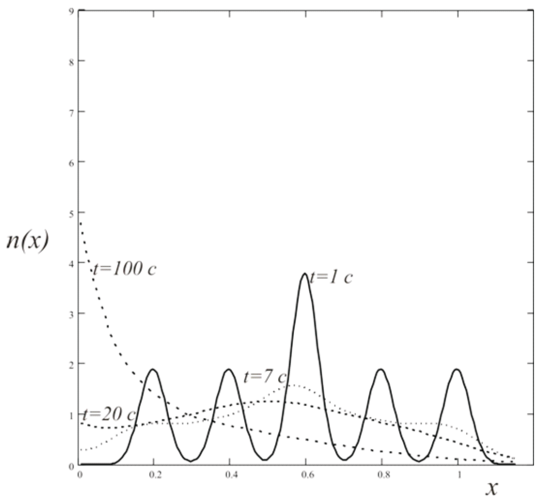

Fig. 8

The dark solid line illustrates the total distribution according to

$$n(t,x) = \frac{1}{N}\sum_{i=1}^{N} n_i(t,x). \tag{21}$$

Fig. 8 shows the dependence of the total distribution when the initial state has uniform distribution of the grains under study at various moments of time.

The resulting distributions enable us, for the prescribed values of the target fractions (for instance, in percent of total volume) with uniform initial distribution, to calculate the likely time that a certain prescribed layer of material takes to form at the bottom of the jigging bed reservoir with a prescribed concentration of the useful fraction.

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