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ON SMOOTH SOLUTIONS TO THE GEVREY PROBLEM FOR THIRD ORDER EQUATIONS

V. I. Antipin and S. V. Popov

Abstract. We consider the Gevrey problem for a forward-backward equation of the third order with multiple characteristics. The agreement conditions are continuous and the theory of integral equations with a homogeneous kernel of degree -1 is employed. Solvability of boundary value problems in Hölder classes is established. It is demonstrated that the Hölder classes of solutions depend on the fulfillment of some necessary and sufficient conditions on the data of the problem.

Keywords: Gevrey problem, forward-backward equation, agreement condition, well-posedness, Hölder space, integral equation with a homogeneous kernel of degree -1

1. Introduction

We consider the Gevrey equation of the third order with multiple characteristics written as

$$u_{xxx} - \operatorname{sgn} x \cdot u_t = F(x,t). \tag{1}$$

Solvability of the boundary value problems for (1) was considered for the first time by T. D. Dzhuraev [1]. As is known, the smoothness of the initial and boundary data ensures the membership of a solution of the usual boundary value problems for strictly parabolic equations in the Hölder space $H_{x\ t}^{p,p/2}$, whereas this is not so for forward-backward equations. In some simplest cases S. A. Tersenov in [2] gives some necessary and sufficient conditions for solvability of the problem in $H_{x\ t}^{p,p/2}$ for $p > 2$. The solvability (orthogonality) conditions for the data of the problem were written down explicitly in [2]. The Gevrey problems are also examined in [3–5]. Note that the number of necessary orthogonality conditions is finite in the one-dimensional case. However, the number of orthogonality conditions (of the integral character) is infinite in the multidimensional case (see [6, 7]). The generalized and regular solvability of the Gevrey problems can be found in [8, 9].

2. Smooth Solvability

Consider (1) in the domain $Q = \Omega \times (0,T)$, $\Omega \equiv \mathbb{R}$. The parts of the strip Q, where $x < 0$ and $x > 0$, are denoted by Q^- and Q^+.

The space $H_{x\ t}^{p,p/3}(Q)$, $p = 3 + \gamma$, $0 < \gamma < 1$, is the Banach space of functions $u(x,t)$ continuous in $\overline{Q}$ together with their derivatives of the form $D_t^r D_x^q$ with $3r + q < p$ having the finite norms

$$|u|_Q^{(p)} = \langle u\rangle_Q^{(p)} + \sum_{j=0}^{3} \sum_{3r+q=j} \left|D_t^r D_x^q u\right|_Q^{(0)}, \quad |u|_Q^{(0)} = \max_Q |u|,$$

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where

$$\langle u\rangle_Q^{(p)} = \langle u\rangle_{x,Q}^{(p)} + \langle u\rangle_{t,Q}^{(p/3)}, \quad \langle u\rangle_{x,Q}^{(p)} = \langle u_t\rangle_{x,Q}^{(\gamma)} + \langle u_{xxx}\rangle_{x,Q}^{(\gamma)},$$

$$\langle u\rangle_{t,Q}^{(p/3)} = \sum_{0<p-3r-q<3} \langle D_t^r D_x^q u\rangle_{t,Q}^{(\frac{p-3r-q}{3})}.$$

We seek for a solution that lies in the Hölder space $H_{x\ t}^{p,p/3}(Q^{\pm})$, $p = 3+\gamma$, $0<\gamma<1$, and satisfies the initial conditions

$$u(x,0) = \varphi_1(x),\ x>0, \quad u(x,T) = \varphi_2(x),\ x<0, \tag{2}$$

and the agreement conditions

$$\frac{\partial^k u}{\partial x^k}(-0,t) = \frac{\partial^k u}{\partial x^k}(+0,t) \quad (k=0,1,2). \tag{3}$$

Uniqueness follows from the arguments similar to those of [1, p. 157].

Existence of a solution. To prove existence, we first write down the fundamental and elementary Cattabriga solutions [10–12] for the equation

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x^3} = 0. \tag{4}$$

These solutions for (4) are of the form

$$U_i(x,t;\xi,\tau) = \begin{cases} \frac{1}{(t-\tau)^{1/3}} f_i\big(\frac{x-\xi}{(t-\tau)^{1/3}}\big), & t>\tau, \\ 0, & t\le\tau, \end{cases} \tag{5}$$

where $f_0(\eta)$ and $f_1(\eta)$, called Airy functions, are linearly independent solutions to the differential equation

$$z''(\eta) + \frac{\eta}{3} z(\eta) = 0 \tag{6}$$

representable as

$$f_0(\eta) = \int_0^\infty \cos(\lambda^3 - \lambda\eta)\, d\lambda, \quad -\infty<\eta<+\infty,$$

$$f_1(\eta) = \int_0^\infty [e^{-\lambda^3-\lambda\eta} + \sin(\lambda^3-\lambda\eta)]\, d\lambda, \quad \eta>-\infty.$$

The fundamental solution $f_0(\eta)$ and the elementary solution $f_1(\eta)$ satisfy the estimates (see [11])

$$\left\{ \begin{array}{l} \big|\frac{\partial^{k+j}}{\partial x^k \partial t^j} U_0(x,t;\xi,\tau)\big| \\ \big|\frac{\partial^{k+j}}{\partial x^k \partial t^j} U_1(x,t;\xi,\tau)\big| \end{array} \right. < C_0 \frac{|x-\xi|^{(2k+6j-1)/4}}{|t-\tau|^{(2k+6j+1)/4}} \tag{7}$$

for $\frac{x-\xi}{(t-\tau)^{1/3}} \to +\infty$, $k+j\ge 1$ and

$$\left|\frac{\partial^{k+j}}{\partial x^k \partial t^j} U_0(x,t;\xi,\tau)\right| < \frac{C_1}{|t-\tau|^{\frac{1+k+3j}{3}}} \exp\left(-C_2 \frac{|x-\xi|^{\frac{3}{2}}}{|t-\tau|^{\frac{1}{2}}}\right) \tag{8}$$

for $\frac{x-\xi}{(t-\tau)^{1/3}} < +\infty$; C_0, C_1, and C_2 are positive constants. Moreover, it is easy to verify that

$$f_0(0) = \frac{\sqrt{3}}{2}\int\limits_0^\infty e^{-\eta^3}\,d\eta = \frac{1}{2\sqrt{3}}\Gamma\Big(\frac{1}{3}\Big), \quad f_1(0) = \frac{3}{2}\int\limits_0^\infty e^{-\eta^3}\,d\eta = \frac{1}{2}\Gamma\Big(\frac{1}{3}\Big),$$

$$f_0'(0) = \frac{\sqrt{3}}{2}\int\limits_0^\infty \eta e^{-\eta^3}\,d\eta = \frac{1}{2\sqrt{3}}\Gamma\Big(\frac{2}{3}\Big), \quad f_1'(0) = -\frac{3}{2}\int\limits_0^\infty \eta e^{-\eta^3}\,d\eta = -\frac{1}{2}\Gamma\Big(\frac{2}{3}\Big), \tag{9}$$

$$\int\limits_0^\infty f_0(\eta)\,d\eta = \frac{2\pi}{3}, \quad \int\limits_{-\infty}^0 f_0(\eta)\,d\eta = \frac{\pi}{3}, \quad \int\limits_0^\infty f_1(\eta)\,d\eta = 0.$$

For convenience, we consider the system

$$u_t^1 = Lu^1, \quad -u_t^2 = Lu^2 \quad \left(L \equiv \frac{\partial^3}{\partial x^3}\right) \tag{10}$$

in Q^+ for $F(x,t) \equiv 0$ rather than (1). The initial and agreement conditions take the form

$$u^1(x,0) = \varphi_1(x), \quad u^2(x,T) = \varphi_2(-x), \ x > 0, \tag{11}$$

$$\frac{\partial^k u^1}{\partial x^k}(0,t) = (-1)^k \frac{\partial^k u^2}{\partial x^k}(0,t) \quad (k = 0,1,2). \tag{12}$$

We assume that $\varphi_i(x) \in H^p(\mathbb{R})$ $(i = 1,2)$. In this case the functions

$$\omega_1(x,t) = \frac{1}{\pi}\int\limits_{\mathbb{R}} U_0(x,t;\xi,0)\varphi_1(\xi)\,d\xi, \quad \omega_2(x,t) = \frac{1}{\pi}\int\limits_{\mathbb{R}} U_0(\xi,T;x,t)\varphi_2(\xi)\,d\xi \tag{13}$$

are solutions to (10) satisfying (11) in $\mathbb{R}$. We use the following representation for solutions to (10):

$$\begin{gathered} u^1(x,t) = \int\limits_0^t U_0(x,t;0,\tau)\alpha_0(\tau)\,d\tau + \int\limits_0^t U_1(x,t;0,\tau)\alpha_1(\tau)\,d\tau + \omega_1(x,t), \\ u^2(x,t) = \int\limits_t^T U_0(0,\tau;x,t)\beta_0(\tau)\,d\tau + \omega_2(x,t). \end{gathered} \tag{14}$$

In view of the general results [11, 13, 14], the densities α_0, α_1, and β_0 have to belong to H^q $(q = \frac{\gamma+1}{3})$, with

$$\alpha_0(0) = \alpha_1(0) = \beta_0(T) = 0. \tag{15}$$

Indeed, note that $u^1(x,t) \in H_{x\ t}^{p,p/3}(Q^+)$, if $\psi_1(t) = u^1(0,t) \in H^{1+\frac{\gamma}{3}}(0,T)$, $\psi_2(t) = u_x^1(0,t) \in H^{\frac{2+\gamma}{3}}(0,T)$ and the consistency conditions hold:

$$\psi_1(0) = \varphi_1(0), \quad \psi_1'(0) = \varphi_1'''(0), \quad \psi_2(0) = \varphi_2'(0). \tag{16}$$

From (9) it follows that

$$\psi_1(t) = f_0(0)\int\limits_0^t \frac{\alpha_0(\tau) + \sqrt{3}\alpha_1(\tau)}{(t-\tau)^{\frac{1}{3}}}\,d\tau + \omega_1(0,t),$$

$$\psi_2(t) = f_0'(0)\int\limits_0^t \frac{\alpha_0(\tau) - \sqrt{3}\alpha_1(\tau)}{(t-\tau)^{\frac{2}{3}}}\,d\tau + \omega_{1x}(0,t).$$

Hence,

$$\psi_1'(t) = f_0(0)\frac{d}{dt}\int\limits_0^t \frac{\alpha_0(\tau)+\sqrt{3}\alpha_1(\tau)-\alpha_0(0)-\sqrt{3}\alpha_1(0)}{(t-\tau)^{\frac{1}{3}}}\,d\tau$$

$$+f_0(0)(\alpha_0(0)+\sqrt{3}\alpha_1(0))t^{-\frac{1}{3}}+\frac{1}{\pi}\int\limits_{\mathbb{R}} U_0(0,t;\xi,0)\varphi_1'''(\xi)\,d\xi, \tag{17}$$

$$\psi_2(t) = f_0'(0)\int\limits_0^t \frac{\alpha_0(\tau)-\sqrt{3}\alpha_1(\tau)-\alpha_0(0)+\sqrt{3}\alpha_1(0)}{(t-\tau)^{\frac{2}{3}}}\,d\tau$$

$$+3f_0'(0)(\alpha_0(0)-\sqrt{3}\alpha_1(0))t^{\frac{1}{3}}+\frac{1}{\pi}\int\limits_{\mathbb{R}} U_0(0,t;\xi,0)\varphi_1'(\xi)\,d\xi.$$

If $\alpha_0(t),\alpha_1(t)\in H^{\frac{1+\gamma}{3}}(0,T)$ then (see [2, 11]):

$$\frac{d}{dt}\int\limits_0^t \frac{\alpha_0(\tau)+\sqrt{3}\alpha_1(\tau)-\alpha_0(0)-\sqrt{3}\alpha_1(0)}{(t-\tau)^{\frac{1}{3}}}\,d\tau \in H^{\frac{\gamma}{3}}(0,T),$$

$$\int\limits_0^t \frac{\alpha_0(\tau)-\sqrt{3}\alpha_1(\tau)-\alpha_0(0)+\sqrt{3}\alpha_1(0)}{(t-\tau)^{\frac{2}{3}}}\,d\tau \in H^{\frac{2+\gamma}{3}}(0,T);$$

moreover,

$$\int\limits_{\mathbb{R}} U_0(0,t;\xi,0)\varphi_1'''(\xi)\,d\xi \in H^{\frac{\gamma}{3}}(0,T),$$

$$\int\limits_{\mathbb{R}} U_0(0,t;\xi,0)\varphi_1'(\xi)\,d\xi \in H^{\frac{2+\gamma}{3}}(0,T).$$

Therefore, under the conditions $\alpha_0(0)=\alpha_1(0)=0$, we see that $\psi_1(t)\in H^{1+\frac{\gamma}{3}}(0,T)$, $\psi_2(t)\in H^{\frac{2+\gamma}{3}}(0,T)$, and the consistency conditions (16) for $\alpha_0(t)$ and $\alpha_1(t)$ hold.

The agreement conditions (12) generate the following system of integral equations with Abel's operators with respect to α_0, α_1, and β_0:

$$\begin{cases} f_0(0)\int\limits_0^t \frac{\alpha_0(\tau)+\sqrt{3}\alpha_1(\tau)}{(t-\tau)^{\frac{1}{3}}}\,d\tau+\omega_1(0,t) = f_0(0)\int\limits_t^T \frac{\beta_0(\tau)}{(\tau-t)^{\frac{1}{3}}}\,d\tau+\omega_2(0,t), \\ f_0'(0)\int\limits_0^t \frac{\alpha_0(\tau)-\sqrt{3}\alpha_1(\tau)}{(t-\tau)^{\frac{2}{3}}}\,d\tau+f_0'(0)\int\limits_t^T \frac{\beta_0(\tau)}{(\tau-t)^{\frac{2}{3}}}\,d\tau+\omega_{1x}(0,t)+\omega_{2x}(0,t)=0, \\ -\frac{2\pi}{3}\alpha_0(t)+\omega_{1xx} = -\frac{\pi}{3}\beta_0(t)+\omega_{2xx}. \end{cases} \tag{18}$$

Equations (18) and the Abel inversion formulas [2] lead to the system of singular integral equations of the form

$$\begin{cases} \frac{2}{\sqrt{3}}(\alpha_0(t)+\sqrt{3}\alpha_1(t))+\frac{1}{\sqrt{3}}\beta_0(t)-\frac{1}{\pi}\int\limits_0^T \left(\frac{\tau}{t}\right)^{2/3}\frac{\beta_0(\tau)}{\tau-t}\,d\tau = \frac{d}{dt}\int\limits_0^t \frac{\Phi_0(\tau)}{(t-\tau)^{2/3}}\,d\tau, \\ \frac{2}{\sqrt{3}}(\alpha_0(t)-\sqrt{3}\alpha_1(t))+\frac{1}{\sqrt{3}}\beta_0(t)+\frac{1}{\pi}\int\limits_0^T \left(\frac{\tau}{t}\right)^{1/3}\frac{\beta_0(\tau)}{\tau-t}\,d\tau = \frac{d}{dt}\int\limits_0^t \frac{\Phi_1(\tau)}{(t-\tau)^{1/3}}\,d\tau, \\ 2\alpha_0(t)-\beta_0(t)=\Phi_2(t), \end{cases} \tag{19}$$

where

$$\Phi_j(t) = \frac{1}{\pi f^{(j)}(0)}\left(\frac{\partial^j \omega_2}{\partial x^j}(0,t) - (-1)^j \frac{\partial^j \omega_1}{\partial x^j}(0,t)\right) \quad (j = 0,1),$$
$$\Phi_2(t) = \frac{3}{\pi}[\omega_{2xx}(0,t) - \omega_{1xx}(0,t)].$$

Put

$$F_0^0(t) = \int_0^t \frac{\Phi_0'(\tau) - \Phi_0'(0)}{(t-\tau)^{\frac{2}{3}}}\, d\tau,\ F_1^0(t) = \frac{d}{dt}\int_0^t \frac{\Phi_1(\tau) - \Phi_1(0)}{(t-\tau)^{\frac{1}{3}}}\, d\tau,\ F_2^0(t) = \Phi_3(t) - \Phi_3(0).$$

Since $\Phi_k^0(t) \in H^{q_k}(0,T)$, $q_k = 1 + \frac{\gamma - k}{3}$ [2], the functions $F_k^0(t)$ $(k = 0,1,2)$ belong to $H^{(1+\gamma)/3}(0,T)$ and $F_k^0(t) = O(t^{(1+\gamma)/3})$ for t small.

Prove the existence of solutions α_1, α_2, and β_0 to (19) belonging to $H^q(0,T)$ $(q = (p-2)/3,\ p = 3 + \gamma,\ 0 < \gamma < 1)$ and satisfying (15).

Assume that α_1, α_2, and β_0 belong to the space. In this case, in view of (19) we have

$$\begin{cases} -\frac{1}{\pi}\int\limits_0^T \frac{\beta_0(\tau)}{\tau^{\frac{1}{3}}}\, d\tau = \Phi_0(0), \\ \frac{1}{\pi}\int\limits_0^T \frac{\beta_0(\tau)}{\tau^{\frac{2}{3}}}\, d\tau = \Phi_1(0), \\ \beta_0(0) = -\Phi_2(0). \end{cases} \tag{20}$$

The last equality is equivalent to the first condition $\alpha_0(0) = 0$ in (15). Note also that the condition $2\alpha_0(T) = \Phi_2(T)$ is equivalent to the equality $\beta_0(T) = 0$.

Under conditions (20) the system of equations (19) is rewritten as

$$\begin{cases} \frac{2}{\sqrt{3}}(\alpha_0(t) + \sqrt{3}\alpha_1(t)) + \frac{1}{\sqrt{3}}\beta_0(t) - \frac{1}{\pi}\int\limits_0^T \left(\frac{t}{\tau}\right)^{1/3} \frac{\beta_0(\tau)}{\tau - t}\, d\tau \\ \qquad\qquad = 3\Phi_0'(0)t^{1/3} + F_0^0(t), \\ \frac{2}{\sqrt{3}}(\alpha_0(t) - \sqrt{3}\alpha_1(t)) + \frac{1}{\sqrt{3}}\beta_0(t) + \frac{1}{\pi}\int\limits_0^T \left(\frac{t}{\tau}\right)^{2/3} \frac{\beta_0(\tau)}{\tau - t}\, d\tau = F_1^0(t), \\ 2\alpha_0(t) - \beta_0(t) + \beta_0(0) = F_2^0(t). \end{cases} \tag{21}$$

In view of the formula [15, p. 177]

$$\frac{1}{\pi}\int_0^T \frac{\tau^{\rho-1}(T-\tau)^{\sigma-1}}{\tau - t}\, d\tau = t^{\rho-1}(T-t)^{\sigma-1}\cot(\sigma\pi)$$
$$-\frac{\Gamma(\rho)\Gamma(\sigma-1)}{\pi\Gamma(\rho+\sigma-1)}T^{\rho+\sigma-2}F\left(2-\rho-\sigma, 1, 2-\sigma; \frac{T-t}{T}\right), \tag{22}$$

we can define $\alpha_1(T)$.

Introduce the new function $\bar{\beta}_0(t) = \beta_0(t) - \beta_0(0)\frac{T-t}{T}$ in (21). Involving (22) we can represent (21) as

$$\begin{cases} \frac{2}{\sqrt{3}}(\alpha_0(t)+\sqrt{3}\alpha_1(t))+\frac{1}{\sqrt{3}}\bar{\beta}_0(t)-\frac{1}{\pi}\int\limits_0^T\left(\frac{t}{\tau}\right)^{1/3}\frac{\bar{\beta}_0(\tau)}{\tau-t}\,d\tau \\ \qquad =-\frac{9}{2\pi}\beta_0(0)F\left(-\frac{2}{3},1,\frac{4}{3};\frac{t}{T}\right)\left(\frac{t}{T}\right)^{\frac{1}{3}}+3\Phi_0'(0)t^{1/3}+F_0^0(t), \\ \frac{2}{\sqrt{3}}(\alpha_0(t)-\sqrt{3}\alpha_1(t))+\frac{1}{\sqrt{3}}\bar{\beta}_0(t)+\frac{1}{\pi}\int\limits_0^T\left(\frac{t}{\tau}\right)^{2/3}\frac{\bar{\beta}_0(\tau)}{\tau-t}\,d\tau \\ \qquad =\frac{9}{2\pi}\beta_0(0)F\left(-\frac{1}{3},1,\frac{5}{3};\frac{t}{T}\right)\left(\frac{t}{T}\right)^{\frac{2}{3}}+F_1^0(t), \\ 2\alpha_0(t)-\bar{\beta}_0(t)=-\beta_0(0)\frac{t}{T}+F_2^0(t). \end{cases} \tag{23}$$

Since $\alpha_0(t)$, $\alpha_1(t)$, and $\beta_0(t)$ must belong to $H^{(1+\gamma)/3}(0,T)$, the first equation of the above system validates the condition

$$-\frac{1}{\pi}\int\limits_0^T\frac{\bar{\beta}_0(\tau)}{\tau^{\frac{4}{3}}}\,d\tau=-\frac{9}{2\pi}\beta_0(0)\frac{1}{T^{\frac{1}{3}}}+3\Phi_0'(0). \tag{24}$$

Under (24), we arrive at the system

$$\begin{cases} \frac{2}{\sqrt{3}}(\alpha_0(t)+\sqrt{3}\alpha_1(t))+\frac{1}{\sqrt{3}}\bar{\beta}_0(t)-\frac{1}{\pi}\int\limits_0^T\left(\frac{t}{\tau}\right)^{4/3}\frac{\bar{\beta}_0(\tau)}{\tau-t}\,d\tau=\overline{F}_0^0(t), \\ \frac{2}{\sqrt{3}}(\alpha_0(t)-\sqrt{3}\alpha_1(t))+\frac{1}{\sqrt{3}}\bar{\beta}_0(t)+\frac{1}{\pi}\int\limits_0^T\left(\frac{t}{\tau}\right)^{2/3}\frac{\bar{\beta}_0(\tau)}{\tau-t}\,d\tau=\overline{F}_1^0(t), \\ 2\alpha_0(t)-\bar{\beta}_0(t)=\overline{F}_2^0(t), \end{cases} \tag{25}$$

where

$$\overline{F}_0^0(t)=-\frac{9}{2\pi}\beta_0(0)\left[F\left(-\frac{2}{3},1,\frac{4}{3};\frac{t}{T}\right)-1\right]\left(\frac{t}{T}\right)^{\frac{1}{3}}+F_0^0(t),$$

$$\overline{F}_1^0(t)=\frac{9}{2\pi}\beta_0(0)F\left(-\frac{1}{3},1,\frac{5}{3};\frac{t}{T}\right)\left(\frac{t}{T}\right)^{\frac{2}{3}}+F_1^0(t),$$

$$\overline{F}_2^0(t)=-\beta_0(0)\frac{t}{T}+F_2^0(t)$$

belong to $H^{(1+\gamma)/3}(0,T)$ and $\overline{F}_j^0(t)=O(t^{\frac{1+\gamma}{3}})$ $(j=0,1,2)$ for small t.

Proceed with the proof of existence of $\alpha_0(t)$, $\alpha_1(t)$, and $\beta_0(t)$ in the system of equations (25) from $H^{(1+\gamma)/3}(0,T)$.

Excluding $\alpha_0(t)$ and $\alpha_1(t)$ from (25), we infer

$$\frac{4}{\sqrt{3}}\bar{\beta}_0(t)+\frac{t^{\frac{2}{3}}}{\pi}\int\limits_0^T K(t,\tau)\bar{\beta}_0(\tau)\,d\tau=Q(t), \tag{26}$$

where

$$K(t,\tau)=\frac{\tau^{\frac{1}{3}}+t^{\frac{1}{3}}}{\tau^{\frac{4}{3}}(\tau^{\frac{2}{3}}+\tau^{\frac{1}{3}}t^{\frac{1}{3}}+t^{\frac{2}{3}})},\quad Q(t)=\overline{F}_0^0(t)+\overline{F}_1^0(t)-\frac{2}{\sqrt{3}}\overline{F}_2^0(t).$$

The kernel $K(t,\tau)$ of (26) satisfies the estimate

$$K(t,\tau)\le\frac{\tau^{\frac{1}{3}}+t^{\frac{1}{3}}}{\tau^{\frac{4}{3}}|\tau-t|^{\frac{2}{3}}} \tag{27}$$

and

$$t^{\frac{2}{3}}K(t,\tau)=\left(\frac{t}{\tau}\right)^{\frac{1+\gamma}{3}}\varphi\left(\frac{t}{\tau}\right)\frac{1}{\tau},\quad \varphi(x)=x^{\frac{1-\gamma}{3}}\frac{1-x^{\frac{2}{3}}}{1-x}.$$

Putting $\beta_1(t) = \bar{\beta}_0(t)t^{-\frac{1+\gamma}{3}}$ and $Q_1(t) = Q(t)t^{-\frac{1+\gamma}{3}}$ in (26), we have

$$\frac{4}{\sqrt{3}}\beta_1(t) + \frac{1}{\pi}\int_0^T \varphi\left(\frac{t}{\tau}\right)\frac{\beta_1(\tau)}{\tau}\,d\tau = Q_1(t). \tag{28}$$

Integral equation (28) has a homogeneous kernel of degree -1 [16]. Introducing the new variables $t = Te^{-y}$, $\tau = Te^{-x}$ and assigning

$$\beta_2(y) = \beta_1(Te^{-y}), \quad Q_2(y) = Q_1(Te^{-y}), \quad h(x) = \varphi(e^{-x}) = e^{(1-\beta)x}K_1(1, e^x),$$
$$K_1(t,\tau) = \tau^{\frac{2}{3}}K(t,\tau), \quad \beta = \frac{1-\gamma}{3},$$

we arrive at the Wiener–Hopf equation (see [16, 17])

$$\frac{4}{\sqrt{3}}\beta_2(y) + \frac{1}{\pi}\int_0^{+\infty} h(y-x)\beta_2(x)\,dx = Q_2(y), \quad 0 < y < +\infty. \tag{29}$$

It is not difficult to justify the integrability condition

$$\int_{-\infty}^{+\infty} |h(x)|\,dx = \int_0^{+\infty} |K_1(1,u)|u^{-\beta}\,du = 2\sqrt{3}\pi\frac{\sin(\beta + \frac{1}{3})\pi}{\sin(3\beta\pi)}$$

for $0 < \beta < \frac{1}{3}$. Hence, (28) is considered in the space $H^{\beta}(0,T)$, $0 < \beta < \frac{1}{3}$. The function

$$h(x) = e^{(1-\beta)x}K_1(1, e^x) = e^{(\frac{1}{6}-\beta)x}\frac{\sinh\frac{x}{3}}{\sinh\frac{x}{2}}$$

is even for $\beta = \frac{1}{6}$. The kernel of (28) is symmetrizable in $E_{\frac{1}{6}}(0,T)$ [16]; moreover, the function (see [16, p. 518])

$$H(x) = 2\int_0^{+\infty}\cos(xt)\frac{\sinh\frac{t}{3}}{\sinh\frac{t}{2}}\,dt = \frac{2\sqrt{3}\pi}{2\cosh(2\pi x) - 1}$$

is positive and monotone on $(0, +\infty)$ and $H(0) = 2\sqrt{3}\pi$. In the space $E_{\frac{1}{6}}(0,T)$ the equation [16]

$$\beta_1(t) + \lambda\int_0^T \varphi\left(\frac{t}{\tau}\right)\frac{\beta_1(\tau)}{\tau}\,d\tau = Q_1(t) \tag{30}$$

is uniquely solvable for $\lambda \in N_\lambda = \left(-\infty; \frac{1}{2\sqrt{3}\pi}\right)$, and $\lambda_0 = -\frac{\sqrt{3}}{4\pi} \in N_\lambda$ for (28).

The study of equations of the form (28) to which Wiener–Hopf theory is inapplicable directly in the Hölder spaces can be found in [19, 20].

The Fredholm property for integral operator (28) follows from Theorem 2 of [19], namely, from the condition that the function

$$B(x) = 1 + \frac{\sqrt{3}}{4\pi}\int_0^{+\infty}\varphi(t)t^{q-ix}\,dt$$

vanishes nowhere on the real axis for all $q \in \mathbb{R}$, which is easy to verify.

Integral equation (26) is examined as an equation with respect to $\beta_3(t) = \bar{\beta}_0(t)t^{-\frac{2}{3}}$. Find solutions $\beta_3(t)$ unbounded at $t = 0$ with singularities of order less than 1 and bounded at $t = T$. Equations (26) yield

$$\frac{4}{\sqrt{3}}\beta_3(t) + \frac{1}{\pi}\int\limits_0^T K(t,\tau)\tau^{\frac{2}{3}}\beta_3(\tau)\,d\tau = \frac{Q(t)}{t^{\frac{2}{3}}}. \tag{31}$$

Equation (31) implies that $\beta_0(T) = 0$ if and only if

$$\frac{1}{\pi}\int\limits_0^T K(T,\tau)\tau^{\frac{2}{3}}\beta_3(\tau)\,d\tau = \frac{Q(T)}{T^{\frac{2}{3}}}. \tag{32}$$

Under condition (32), we obtain the equation

$$\frac{4}{\sqrt{3}}\beta_3(t) + \frac{1}{\pi}\int\limits_0^T K_2(t,\tau)\beta_3(\tau)\,d\tau = Q_3(t), \tag{33}$$

where

$$K_2(t,\tau) = \frac{(T^{\frac{1}{3}} - t^{\frac{1}{3}})(\tau^{\frac{1}{3}}T^{\frac{1}{3}} + \tau^{\frac{1}{3}}t^{\frac{1}{3}} + t^{\frac{1}{3}}T^{\frac{1}{3}})}{\tau^{\frac{2}{3}}(\tau^{\frac{2}{3}} + \tau^{\frac{1}{3}}t^{\frac{1}{3}} + t^{\frac{2}{3}})(\tau^{\frac{2}{3}} + \tau^{\frac{1}{3}}T^{\frac{1}{3}} + T^{\frac{2}{3}})}, \quad Q_3(t) = \frac{Q(t)}{t^{\frac{2}{3}}} - \frac{Q(T)}{T^{\frac{2}{3}}}.$$

The function $K_3(t,\tau) = \frac{K_2(t,\tau)}{T^{\frac{1}{3}} - t^{\frac{1}{3}}}$ satisfies the estimates

$$0 \le K_3(t,\tau) \le \frac{\tau^{\frac{1}{3}}T^{\frac{1}{3}} + \tau^{\frac{1}{3}}t^{\frac{1}{3}} + t^{\frac{1}{3}}T^{\frac{1}{3}}}{\tau^{\frac{2}{3}}|\tau - t|^{\frac{2}{3}}|\tau - T|^{\frac{2}{3}}}. \tag{34}$$

We can easily derive that the functions $K_3(t,\tau)t^{\frac{2}{3}}$ and $Q_3(t)t^{\frac{2}{3}}$ at the endpoints 0 and T behave as $t^{\frac{1+\gamma}{3}}(T-t)^{\frac{1+\gamma}{3}}$ and $t^{\frac{1+\gamma}{3}}(T-t)^{\frac{1+\gamma}{3}}$; moreover, [19, Section 51] $\beta_0(t) \in H^{\frac{1+\gamma}{3}}(0,T)$ and $\alpha_k(t) \in H^{\frac{1+\gamma}{3}}(0,T)$ $(k = 0,1)$.

The behavior of the integral $\frac{1}{\pi}\int_0^T K_3(t,\tau)\beta_3(\tau)\,d\tau$ at the endpoints of the integration contour is defined (see [13, p. 136]) by the formula

$$\frac{1}{\Gamma(\sigma)}\int\limits_0^t \tau^{\rho-1}(t-\tau)^{\sigma-1}\,d\tau = \frac{\Gamma(\rho)}{\Gamma(\rho+\sigma)}t^{\rho+\sigma-1}.$$

It is easy to find that

$$\frac{1}{\pi}\int\limits_t^T K_3(t,\tau)\beta_3(\tau)\,d\tau = O((T-t)^{\frac{\gamma}{3}}) \quad \text{for small } T-t,$$

$$\frac{t^{\frac{2}{3}}}{\pi}\int\limits_0^t K_3(t,\tau)\beta_3(\tau)\,d\tau = O(t^{\frac{1+\gamma}{3}}) \quad \text{for small } t.$$

The systems of equations (25) are equivalent to (18) under the four conditions (20), (24), and (32). Inserting values of $\alpha_0(t)$, $\alpha_1(t)$, and $\beta_0(t)$ into (20), (24), and (32), we justify four solvability conditions of (1)–(3) in $H^{p,p/3}_{x\ t}(Q)$. These conditions are denoted by

$$L_s(\varphi_1,\varphi_2) = 0, \quad s = 1,2,3,4. \tag{35}$$

Thus, we have proven the following theorem.

Theorem. *Let $\varphi_1, \varphi_2 \in H^p$, $p = 3 + \gamma$, $0 < \gamma < 1$. Then under the four conditions* (35), *there is a unique solution to* (1) *belonging to Q from $H_{x\ t}^{p,p/3}(Q^{\pm})$ and satisfying* (2) *and* (3).

REMARK. Similar studies can be fulfilled in the case of $\varphi_1, \varphi_2 \in H^p$ ($p = 3l + \gamma$), $0 < \gamma < 1$, where $l \geq 1$ is an integer.

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V. I. Antipin; S. V. Popov
North-Eastern Federal University
Institute of Mathematics, Yakutsk, Russia
madu@ysu.ru; antvasiv@mail.ru

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ON THE DIFFERENTIAL GEOMETRY OF FIVE–DIMENSIONAL B–COMPLEXES OF PLANES IN THE PROJECTIVE SPACE P^5

I. V. Bubyakin

Abstract. The differential geometry of five-dimensional B-complexes is considered in the projective space P^5. The structure is determined five-dimensional B-complexes of two-dimensional planes.

Keywords: complexes of two-dimensional plane, Grassmann map, Segre manifold

1. The article deals with that part of the multidimensional projective differential geometry which is devoted to the studies of families of planes of various dimensions in the projective space. Many articles by Soviet, Russian, and foreign geometricians are devoted to this theory. It was shaped in the articles on the line congruence theory and the theory of complexes of lines. These studies were collected in the celebrated monographs by Finikov [1] and Kovantsev [2], respectively. Afterwards, the method of exterior Cartan forms [3] made it possible to develop these theories in a more general situation for families of m-dimensional planes of an arbitrary dimension. Many questions are of interest for not only multidimensional differential geometry but the Radon–Helgason integral geometry as well which is a new direction in the modern mathematics (see the monograph [4] by Gelfand, Gindikin, and Graev) and the general theory of partial differential equations, for example, as exposed in the article [5] by Gelfand and Graev, where the hypergeometric functions connected with the Grassmann manifold of two-dimensional planes in a five-dimensional space are examined. In integral geometry the complexes of m-dimensional planes of the same dimension as the space itself are considered when solving the main problem.

One of the most beautiful fields of differential geometry, where the advantages of coordinate-free methods are fully exhibited, is the theory of complexes of multidimensional planes of the projective space. This interest in the theory of complexes of multidimensional planes is stipulated also by the problems of integral geometry in which we need to recover a function from its integrals over planes of some family. The main problem is to describe the so-called admissible complexes for which the recovering is possible. To solve this problem successfully, it is necessary to combine the methods of integral geometry with various beautiful constructions within the framework of the projective geometry of complexes of multidimensional planes. At the same time the differential and geometric studies of admissible complexes of planes which are of great importance in integral geometry stand aside.

Hence, the projective differential geometry of complexes of multidimensional planes seems topical. The admissible five-dimensional complexes of two-dimensional planes in the projective space P^5 are the object of the study of this article. The complexes of two-dimensional planes are generalizations of complexes of straight lines in the three-dimensional space in the sense that the two-dimensional planes in a five-dimensional space and straight lines in a three-dimensional space are self-dual. The family of two-dimensional planes p in the five-dimensional projective space P^5 is also self-dual, since its image under a correlative transformation is a family of the same type. In view of this fact, all constructions connected with such a family admits a dual interpretation. The dual constructions allows us to conduct the studies without any additional arguments. These constructions are widely used in the differential geometry of families of complexes of two-dimensional planes [6].

For the first time, some generalizations of admissible complexes of straight lines of the projective space and their geometric structure were examined by Kruglyakov in [7, 8] (they are called K-admissible complexes), Vasil′ev, and Nersesyan in [9, 10] (they are called N-admissible complexes), and also by Maius, the Hungarian geometrician, and Goncharov [14–16] (these complexes are called admissible in the integral geometry sense or just admissible complexes of plains). The constructions of K-admissible complexes of two-dimensional planes coincide with some of the constructions of admissible complexes of two-dimensional planes from [14–16] and [11–13], and the constructions of N-admissible complexes of planes are different from those in the integral geometry sense.

In an n-dimensional projective space P^n, Vasil′ev and Nersesyan [9, 10] call the n-dimensional complexes C^n of m-dimensional planes *admissible* (N*-admissible*) whenever the following holds: Given a point M in an arbitrary generator $p \in C^n$, the containing tangent plane to the cone formed by the planes of the complex passing through M is independent of the choice of $M \in C^n$. Kruglyakov [7, 8] call the n-dimensional complexes C^n of m-dimensional planes *admissible* (K*-admissible*) if the following holds: For an $(m-1)$-dimensional plane p^{m-1} with an arbitrary generator $p \in C^n$, the containing tangent plane to the cone formed by the planes of the complex passing through $(m-1)$-dimensional plane p^{m-1} is independent of the choice of the $(m-1)$-dimensional plane $p^{m-1} \in C^n$. The Grassmann mapping [17] is a bijective mapping of the manifold $G(2,5)$ of two-dimensional planes in the projective space P^5 onto a nine-dimensional point algebraic manifold $\Omega(2,5)$ from the projective space P^{19}. The tangent space $T_p\Omega(2,5)$ to the manifold $\Omega(2,5)$ in an arbitrary point p contains the five-dimensional asymptotic cone $B_p(2)$ [18] connected with a neighborhood of the second order whose projectivization is the Segre manifold $S_p(2,2)$. The Segre manifold $S_p(2,2)$ remains invariant under the projective transformations of the space $P^8 = PT_p\Omega(2,5)$, which is the projectivization centered at p of the tangent space $T_p\Omega(2,5)$ to $\Omega(2,5)$. Moreover, $T_p\Omega(2,5)$ contains the eight-dimensional asymptotic cone $B_p(3)$ [18] connected with a neighborhood of the third order whose projectivization is the cubic hypersurface $PB_p(3)$ in P^8.

To a five-dimensional complex C^5 of two-dimensional planes on the algebraic manifold $\Omega(2,5)$, there corresponds a five-dimensional smooth manifold V^5. The study of the mutual location of the four-dimensional tangent plane PT_pV^5 which is the projectivization centered at a point p of the tangent plane T_pV^5 to the manifold V^5 with the Segre manifold $S_p(2,2)$ and the cubic hypersurface $PB_p(3)$ gives us a possibility of distinguishing N- and K-admissible compact sets with a common viewpoint. Namely, as is exhibited in [6], N-admissible complexes are characterized by the fact that for every two-dimensional generator p the intersection of the four-dimensional plane PT_pV^5 and the hypercube $PB_p(3)$ is the three-dimensional cubic surface Q^3 which decomposes into the cone Q^2 of second order and the three-dimensional plane from the α-generator of the hypercube $PB_p(3)$ while K-admissible complexes are characterized by the fact that for every two-dimensional generator p the four-dimensional plane PT_pV^5 contains the α-generator of the Segre manifold $S_p(2,2)$. Moreover, this methodological approach allows us also to distinguish some five-dimensional complexes of two-dimensional planes which are admissible complexes in the integral geometry sense.

Thereby, we arrive at the problem of a generalization of the notion of admissible complex of straight lines in the projective space P^n on the base of a map of the Grassmann manifold $G(m,n)$ onto the algebraic manifold $\Omega(m,n)$ of P^N, with $N = C_{n+1}^{m+1} - 1$. Given n-dimensional admissible complexes of m-dimensional planes in P^n, the problem is to find their full geometrical description and structure. Full geometrical description of these complexes can be used in multidimensional differential geometry and integral geometry as well. It is natural to begin these studies with generalization of the complex of straight lines in the three-dimensional projective space, namely, with five-dimensional complexes of two-dimensional planes in P^5.

2. In P^5 each two-dimensional plane p is defined by three linearly-independent points. The matrix of coordinates of these points is used to define $\begin{pmatrix} 3 \\ 6 \end{pmatrix} = 20$ third order determinants that are called *Grassmann coordinates* of p. They are connected by a system of algebraic equations and give rise to the bijective mapping of the Grassmann manifold $G(2,5)$ of two-dimensional planes of the space P^5 onto a nine-dimensional algebraic manifold $\Omega(2,5)$ of the projective space P^{19}. This mapping is called the *Grassmann mapping* [17].

Let us study the structure of $\Omega(2,5)$ in more detail. Consider two two-dimensional planes in the space P^5 whose intersection is a straight line. They generate a linear pencil of planes, i.e. a family of two-dimensional planes containing a line and lying in a three-dimensional plane. To this linear pencil on $\Omega(2,5)$, there corresponds a rectilinear generator. In this case the straight line and the three-dimensional plane containing this line completely define the linear pencil and so the straight line on $\Omega(2,5)$.

Consider all two-dimensional planes lying in a fixed three-dimensional plane. They form a linear three-parameter family to which on the manifold $\Omega(2,5)$ there corresponds some three-dimensional flat generator, the so-called *α-generator*. Since P^5 contains eight-parameter family of three-dimensional planes, $\Omega(2,5)$ carries a family of α-generators depending on eight parameters.

Fix a straight line in P^5. Consider all two-dimensional planes passing through the line. These two-dimensional planes generate a three-parameter bundle to which on $\Omega(2,5)$ there corresponds some three-dimensional flat generator, the so-called β-generator. Since the space P^5 contains an eight-parameter family of straight lines, the manifold $\Omega(2,5)$ carries a family of β-generators depending on eight parameters. Thus, $\Omega(2,5)$ carries the two families of three-dimensional flat generators.

If a three-dimensional plane of P^5 contains a fixed straight line then the intersection of the corresponding α- and β-generators, forming $\Omega(2,5)$, is a straight line. If a three-dimensional plane in P^5 does not contain a straight line then the corresponding flat generators of $\Omega(2,5)$ are disjoint.

Consider a fixed two-dimensional plane p in P^5. A family of two-parameter three-dimensional planes passes through this plane. Hence, a family of two-parameter α-generators passes through the point on $\Omega(2,5)$ corresponding to p. At the same time p contains a two-parameter family of straight lines. Hence, a two-parameter family of β-generators of the manifold passes through p. In this case two generators of different families of $\Omega(2,5)$ passing through p have a common line to which in P^5 there corresponds some linear pencil of two-dimensional planes and two generators of one family have only one common point p. Therefore, all three-dimensional flat generators passing through p are flat generators of the Segre cone $C_p(3,3)$ [18] with vertex p on the manifold $\Omega(2,5)$. This cone is the intersection of the tangent plane $T_p\Omega(2,5)$ at p to $\Omega(2,5)$ with the manifold itself. In P^5 to the Segre cone $C_p(3,3)$ there corresponds the collection of two-dimensional planes whose intersections with the two-dimensional plane p are straight lines.

Given the projective space P^5, examine a five-parameter family of two-dimensional planes, i.e. a five-dimensional complex C^5. To the complex C^5 under the Grassmann map [17–19] there corresponds the five-dimensional manifold V^5 which belongs to the algebraic manifold $\Omega(2,5)$. The manifold V^5 at every its point p has five-dimensional tangent plane T_pV^5. The projectivization of T_pV^5 centered at p is the four-dimensional plane PT_pV^5. To different mutual locations of the plane PT_pV^5 and the Segre manifold $S_p(2,2)$ there correspond different classes of complexes C^5 [17–19]. The Segre manifold $S_p(2,2)$ is a four-dimensional algebraic surface of sixth order carrying two two-parameter families of two-dimensional flat generators. In this case two generators of different families have a common point and two generators from one family are disjoint. Since the Segre manifold $S_p(2,2)$ is an algebraic surface of the sixth order, in the general case the intersection of the manifold and the plane PT_pV^5 consists of sixth points. These points define the sixth fields of directions on V^5 to whose integral lines on the complex C^5 there correspond six families of torses (developable surfaces with two-dimensional flat generators) [6] formed by two-dimensional planes osculating with some curve. Six torses (one from each family) pass through every generator. Every torse of the complex defines, given a two-dimensional generator p of the complex C^5, the characteristic straight line (the intersection of two infinitely close generators of the torse), and the three-dimensional characteristic plane (tangent to the torse).

3. Given the projective space P^5, consider a family of point frames $\{A_I\}$, $I = 0, 1, \ldots, 5$, and a family of frames formed by the hyperplanes $\alpha^I = (-1)^I(A_0, \ldots, A_{I-1}, A_{I+1}, \ldots, A_5)$. The equations of motion of these frames are of the form

$$dA_I = \omega_I^J A_J, \quad d\alpha^I = \omega_J^I \alpha^J,$$

where ω_I^J are linear differential forms satisfying the structure equations of P^5, i.e.,

$$d\omega_I^J = \omega_I^K \wedge \omega_K^J, \quad I, J, K = 0, 1, \ldots, 5.$$

Related to a two-dimensional plane p of the space P^5 a family of point frames so that A_i, $i = 0, 1, \ldots, 5$, belong to p. In this case

$$dA_i = \omega_i^j A_j + \omega_i^p A_p, \quad dA_p = \omega_p^i A_i + \omega_i^q A_q,$$

where $i, j = 0, 1, 2$ and $p, q = 3, 4, 5$. Hence, we see that the two-dimensional plane p in P^5 depends on nine parameters and the forms ω_p^i are spanned by their differentials.

Let ω_p^i, $i = 0, 1, 2$, $p = 3, 4, 5$, be linear differential forms specifying the motion of the plane $p = A_0 \wedge A_1 \wedge A_2$ in P^5. Since the dimension of the complex C^5 in question is equal to five, the following four linearly independent equations hold on C^5:

$$\Lambda_p^{\alpha i} \omega_i^p = 0, \tag{1}$$

where $\alpha = 1, 2, 3, 4$. These equations define the four-dimensional plane PT_pV^5 in $P^8 = PT_p\Omega(2,5)$.

A one-parameter family of two-dimensional planes p is a three-dimensional surface with two-dimensional flat generators. This surface is a torse [20] if it is tangentially degenerate of rank one. To a torse on $\Omega(2,5)$, there corresponds the curve whose tangents serve as rectilinear generators of this surface. This curve coincides with the asymptotic line of $\Omega(2,5)$. Hence,

$$\operatorname{rang}\left(\omega_i^p\right) = 1 \tag{2}$$

at every point of this line. Thus, the equation of a torse in the space can be written parametrically as follows:

$$\omega_i^p = \alpha_i x^p \, dt.$$

On $\Omega(2,5)$, the asymptotic directions of second order emerging from the point p are defined from the condition

$$D^2 p = 0 (\operatorname{mod} T_p\Omega(2,5)),$$

which implies that the equation of the cone $B_p(2)$ of asymptotic directions of second order are of the form

$$\omega_i^p \omega_j^q - \omega_i^q \omega_j^p = 0.$$

These equations imply that the coordinates ω_i^p of a point in $B_p(2)$ satisfy condition (2) and so they admit the asymptotic representation

$$\omega_i^p = \alpha_i x^p.$$

Hence, the cone $B_p(2)$ of second order asymptotic directions coincides with the Segre cone $C_p(3,3)$.

The third order asymptotic directions of the manifold $\Omega(2,5)$ emerging from p are specified by the condition

$$d^3 p = 0\big(\operatorname{mod} T_p^3\Omega(2,5)\big), \tag{3}$$

where

$$d^3 p = 6 \det\left(\omega_i^p\right) A_3 \wedge A_4 \wedge A_5 \big(\operatorname{mod} T_p^2\Omega(2,5)\big).$$

Consider the projectivization of the tangent plane $T_p\Omega(2,5)$ centered at p which is the projective space $P^8 = PT_p\Omega(2,5)$, where ω_i^p are homogeneous coordinates of a point. The third order asymptotic directions of the manifold $\Omega(2,5)$ form a cone with the vertex at p denoted by $B_p(3)$. In view of (3), $B_p(3)$ is defined by the equation

$$\det\left(\omega_i^p\right) = 0. \tag{4}$$

Hence, $B_p(3)$ is a hypercone of third order in the tangent plane $T_p\Omega(2,5)$ at p to the manifold $\Omega(2,5)$.

The geometric sense of $B_p(3)$ is described as follows: Each hyperplane in P^5 passing through p contains a sixth-parameter family of two-dimensional planes to which on the algebraic manifold $\Omega(2,5)$ there corresponds the submanifold $\Omega(2,4)$ passing through p. The six-dimensional tangent planes to these submanifolds constitute a family of flat generators of the cone $B_p(3)$ which are called α-generators. A sixth-parameter family of two-dimensional planes passes through every point of the plane p; to this family there corresponds some submanifold $\Omega^*(2,4)$ on the manifold $\Omega(2,5)$ also passing through p. Six-dimensional tangent planes to these submanifolds form the second family of flat generators of the cone $B_p(3)$ which are called its β-generators. Thus, the cone $B_p(3)$ carries two families of six-dimensional flat generators. It follows from (3) that the six-dimensional subspace defined in the space $T_p\Omega(2,5)$ by the equations

$$\alpha_p \omega_i^p = 0$$

belongs to the asymptotic cone $B_p(3)$. This subspace coincides with α-generators of $B_p(3)$. The six-dimensional subspace defined in the space $T_p\Omega(2,5)$ by the equations

$$\beta^i \omega_i^p = 0$$

also belongs to the asymptotic cone $B_p(3)$. It coincides with the β-generators of $B_p(3)$. As is easily seen, the intersection of two generators of different families of the cone $B_p(3)$ is a four-dimensional plane to which in P^5 there corresponds the set of two-dimensional planes passing through some point and belonging to a fixed hyperplane and the intersection of two generators of the same family is a three-dimensional plane being a generator of $B_p(2)$ of second order asymptotic directions.

To the asymptotic cone $B_p(3)$ in $P^8 = PT_l\Omega(2,5)$ there corresponds the cubic hypersurface $PB_p(3)$ defined by the same equation (4) as $B_p(3)$ in the tangent space $T_p\Omega(2,5)$. The hypercube $PB_p(3)$ carries a family of α-generators obtained by the projectivization centered at p of α-generators of the cone $B_p(3)$ and the family of β-generators obtained by the projectivization of β-generators of the cone $B_p(3)$. Note that the Segre manifold $S_p(2,2)$ is the set of double points of the hypercube $PB_p(3)$. The intersection of the plane PT_pV and the hypercube $PB_p(3)$ is generally a three-dimensional cubic surface Q_3 carrying two two-dimensional families of rectilinear generators and two generators of different families pass through every point of this surface.

4. The five-dimensional complexes C^5 in the projective space P^5 can be defined as the intersection of four hypercomplexes of one bundle. As a result of the above frame specialization, the equation of a bundle μ of the hypercomplexes C^8 of two-dimensional planes p in P^5 is written as

$$\mu_\alpha \wedge {}^{\alpha i}_{p} \omega_i^p = 0, \tag{5}$$

where $i = 0,1,2$, $p = 3,4,5$, $\alpha = 1,2,3,4$, and ω_i^p are linear differential forms whose vanishing fixes a two-dimensional plane p on the five-dimensional complex C^5. The projectivization of this bundle of complexes is the three-dimensional projective space P^{*3} whose homogeneous coordinates are the coefficients μ_α of the bundle of hypercomplexes.

Consider the hypercomplex C^8 defined by (3) for some fixed values of the coefficients μ_α. A two-parameter family of two-dimensional planes p of the hypercomplex C^8 forming a hypercone with the vertex p^1 passes through every straight line $p^1 \subset p$. The tangent hyperplanes to these cones in the general case intersect on a two-dimensional plane p, i.e.,

$$\mu_\alpha \wedge {}^{\alpha i}_{p} x^p = 0.$$

Under the conditions

$$\operatorname{rang}\left(\mu_\alpha \wedge {}^{\alpha i}_{p}\right) = 1, \tag{6}$$

(5) defines a hyperplane tangent to the hypercone of two-dimensional planes p with a one-dimensional vertex $p^1 \subset p$. Each three-dimensional plane lying in this tangent hyperplane is a tangent three-dimensional plane to a torse belonging to C^5.

Let us proceed with dual constructions. Each three-dimensional plane $p^3 \supset p$ contains a two-parameter family of two-dimensional planes p of the hypercomplex C^8 whose envelope is a two-dimensional tangentially nondegenerate surface; i.e., at every three-dimensional plane p^3 there exists a point describing a two-dimensional tangentially nondegenerate surface. The manifold of all these points is given by the system of equations

$$\mu_\alpha \wedge_p^{\alpha i} a_i = 0.$$

Under condition (6), this system of equations defines a point at the center of the pencil of straight lines lying in the two-dimensional generator p of the hypercomplex C^8; each straight line of this pencil is a characteristic line of a torse belonging to C^5.

Condition (6) defines in the three-dimensional projective space P^{*3} the intersection of four linearly independent quadratic surfaces Q_α which generally have no common points. Consider the five-dimensional complexes C^5 of two-dimensional planes p defined by the bundle of hypercomplexes C^8 with the property that under the Grassmann map hyperplanes of the bundle of hyperplanes PT_pV^8 contain only one α-generator of the cubic surface $PB_p(3)$ and the intersection of the four-dimensional plane PT_pV^5 with the Segre manifold $S_p(2,2)$ contains two straight lines from different α-generators of $S_p(2,2)$ and one straight line in the β-generator of $S_p(2,2)$. We call chosen five-dimensional complexes C^5 of two-dimensional planes p B-complexes. Note that five-dimensional B-complexes of two-dimensional planes are admissible [14–16].

The choice of the above complexes C^5 of two-dimensional planes p leads to four hypercomplexes for which the corresponding hyperplanes PT_pV^8 under the Grassmann map contain α-generators of the hypercube $PB_p(3)$ and thus

$$\omega_0^5 = 0, \quad \omega_1^4 = 0, \quad \omega_2^3 = 0, \quad \omega_1^5 - \omega_2^5 = 0. \tag{7}$$

The equation of the bundle μ of hypercomplexes C^8 of two-dimensional planes p in this case is written as

$$\alpha\omega_0^5 + \beta\omega_1^4 + \gamma\omega_2^3 + \delta\big(\omega_1^5 - \omega_2^5\big) = 0.$$

The center of this bundle μ of the hypercomplexes C^8 is a five-dimensional B-complex C^5 of two-dimensional planes p being the intersection of four above hypercomplexes C^8 and defined by the system of four differential equations (7).

It is easy to verify that the intersection of the four-dimensional plane PT_pV^5 with the Segre manifold $S_p(2,2)$ contains the two straight lines

$$\omega_0^4 = 0, \quad \omega_2^4 = 0, \quad \omega_1^5 = 0, \tag{8}$$

$$\omega_0^3 = 0, \quad \omega_1^3 = 0, \quad \omega_1^5 = 0 \tag{9}$$

from two different α-generators of $S_p(2,2)$ and one straight line

$$\omega_1^3 = 0, \quad \omega_2^4 = 0, \quad \omega_1^5 = 0 \tag{10}$$

from a β-generator of $S_p(2,2)$. Now we clarify the structure of the five-dimensional B-complexes C^5 of two-dimensional planes p.

Theorem 1. *The five-dimensional B-complexes C^5 are a manifold of two-dimensional planes belonging to hyperplanes of one-parameter family, being tangent at every hyperplane of this family to three-dimensional tangentially nondegenerate surfaces.*

Proof. A B-complex C^5 is defined by the differential equations (7) and the forms ω_0^3, ω_0^4, ω_1^3, ω_2^4, and ω_1^5 are linearly independent on it. Hence, we can take the latter as basis forms for the B-complex C^5. Applying the exterior derivation to (7), we arrive at the quadratic equations

$$\begin{gathered}\omega_3^5 \wedge \omega_0^3 + \omega_4^5 \wedge \omega_0^4 + \left(\omega_0^1 + \omega_0^2\right) \wedge \omega_1^5 = 0, \\ \omega_1^0 \wedge \omega_0^4 - \omega_3^4 \wedge \omega_1^3 + \omega_1^2 \wedge \omega_2^4 - \omega_5^4 \wedge \omega_1^5 = 0, \\ \omega_2^0 \wedge \omega_0^3 + \omega_2^1 \wedge \omega_1^3 - \omega_4^3 \wedge \omega_2^4 - \omega_5^3 \wedge \omega_1^5 = 0, \\ \omega_3^5 \wedge \omega_1^3 + \omega_4^5 \wedge \omega_2^4 + \left(\omega_1^1 - \omega_2^2 + \omega_1^2 - \omega_2^1\right) \wedge \omega_1^5 = 0.\end{gathered} \tag{11}$$

The first quadratic equation of (11) implies that the forms ω_3^5 and ω_4^5 are expressed through the basis forms ω_0^3, ω_0^4, and ω_1^5 of the B-complex C^5 and the last quadratic equation of the that the same forms are expressed through the basis forms ω_1^3, ω_2^4, and ω_1^5. In view of uniqueness of the decomposition of forms on the basis of the B-complex C^5 we obtain that these forms are expressed through only one basis form ω_1^5, i.e., we have the equations

$$\omega_3^5 = a\omega_1^5, \quad \omega_4^5 = b\omega_1^5. \tag{12}$$

The differential of the hyperplane $A_0 \wedge A_1 \wedge A_2 \wedge A_3 \wedge A_4$ due to these equations is written as

$$\begin{aligned} d(A_0 \wedge A_1 \wedge A_2 \wedge A_3 \wedge A_4) &= \left(\omega_0^0 + \omega_1^1 + \omega_2^2 + \omega_3^3 + \omega_4^4\right)(A_0 \wedge A_1 \wedge A_2 \wedge A_3 \wedge A_4) \\ &- \omega_1^5\{(A_0 \wedge A_2 \wedge A_3 \wedge A_4 \wedge A_5) - (A_0 \wedge A_1 \wedge A_3 \wedge A_4 \wedge A_5) \\ &+ a\ (A_0 \wedge A_1 \wedge A_2 \wedge A_4 \wedge A_5) - b(A_0 \wedge A_1 \wedge A_2 \wedge A_3 \wedge A_5)\}. \end{aligned} \tag{13}$$

Hence, the hyperplane $A_0 \wedge A_1 \wedge A_2 \wedge A_3 \wedge A_4$ describes a one-parameter family with a three-dimensional characteristic plane defined by the equations

$$x^1 + x^2 + a\ x^3 + bx^4 = 0. \tag{14}$$

Insert in this characteristic plane of the one-parameter family of hyperplanes $A_0 \wedge A_1 \wedge A_2 \wedge A_3 \wedge A_4$ the vertices A_0, A_1, A_2, A_3, and A_4 of the moving frame. In view of this specialization of a moving frame, we infer

$$a = b = 0, \tag{15}$$

and the equation (14) of the three-dimensional characteristic plane takes the form

$$x^1 + x^2 = 0.$$

In the fixed hyperplane $A_0 \wedge A_1 \wedge A_2 \wedge A_3 \wedge A_4$, i.e., for $\omega_1^5 = 0$, we find that the two-dimensional planes p of the B-complex C^5 are tangent to two three-dimensional tangentially nondegenerate surfaces given by the equations

$$\omega_1^4 = 0, \quad \omega_2^3 = 0. \tag{16}$$

In this case the points A_1 and A_2 at every hyperplane $A_0 \wedge A_1 \wedge A_2 \wedge A_3 \wedge A_4$ of the one-parameter family specify three-dimensional tangentially nondegenerate surfaces whose tangent 3-planes coincide with the respective three-dimensional planes $A_0 \wedge A_1 \wedge A_2 \wedge A_3$ and $A_0 \wedge A_1 \wedge A_2 \wedge A_4$.

Thus, the two-dimensional generators p of a five-dimensional B-complex C^5 belong to hyperplanes of a one-parameter family and are tangent at every hyperplane of this family with two three-dimensional tangentially nondegenerate surfaces.

Let us prove the converse. Consider the set of two-dimensional planes p belonging to hyperplanes of the one-parameter family which are tangent at every hyperplane of this family to two three-dimensional tangentially nondegenerate surfaces. Insert the vertices A_0, A_1, A_2, A_3, and A_4 of the moving frame $\{A_I\}$ in a hyperplane of the one-parameter family and the points A_0, A_1, A_2, A_3, and A_4 in the three-dimensional characteristic plane of this family. We superpose the vertices A_1 and A_2 with the current points of the three-dimensional tangentially nondegenerate surfaces. Insert the points A_0, A_1, and A_2 on the two-dimensional plane presenting the intersection of three-dimensional tangent planes to the tangentially nondegenerate surfaces and arrange A_3 and A_4 in the three-dimensional tangent planes to the tangentially nondegenerate surfaces. In view of the above specialization of the moving frame, the one-parameter family of hyperplanes $A_0 \wedge A_1 \wedge A_2 \wedge A_3 \wedge A_4$ is defined by the following equations:

$$\omega_0^5 = 0, \quad \omega_1^5 - \omega_2^5 = 0, \quad \omega_3^5 = 0, \quad \omega_4^5 = 0, \tag{17}$$

where ω_1^5 is a basis form on this family of hyperplanes.

The three-dimensional tangentially nondegenerate surfaces lying at every hyperplane $A_0 \wedge A_1 \wedge A_2 \wedge A_3 \wedge A_4$ of the one-parameter family of hyperplanes are given by (16) due to the specialization of the moving frame. From (16) and (17) we obtain that (7) define a five-dimensional complex C^5 of two-dimensional planes p. It is not difficult to establish that such complexes are defined by a bundle of hypercomplexes C^8 such that under the Grassmann map the hyperplanes belonging to the bundle PT_pV^8 of hyperplanes contain only one α-generator of the cubic surface $PB_p(3)$ and the intersection of the three-dimensional plane PT_pV^5 with the Segre manifold $S_p(2,2)$ contains two straight lines from two different α-generators of $S_p(2,2)$, defined by equations (8) and (9) and one straight line from a β-generator of the manifold $S_p(2,2)$ defined by the equation (10), i.e., they are five-dimensional B-complexes of two-dimensional planes p. Thus, the conjecture about the structure of the five-dimensional B-complexes of two-dimensional planes p is completely proven.

We can conduct dual constructions; i.e., we can take β-generators of the cubic surface $PB_p(3)$ in the definition of B-complexes C^5 of two-dimensional planes p in the projective space P^5. Consider five-dimensional complexes C^5 of two-dimensional planes p in the projective space P^5 defined by a bundle μ of the hypercomplexes C^8 such that under the Grassmann map the hyperplanes of the bundle PT_pV^8 of hyperplanes contain only one β-generator of the hypercube $PB_p(3)$ and the intersection of the four-dimensional plane PT_pV^5 with the Segre manifold $S_p(2,2)$ contains two straight lines belonging to different β-generators of $S_p(2,2)$ and one straight line belonging to an α-generator of the manifold $S_p(2,2)$. We call such five-dimensional complexes C^5 of two-dimensional planes p *dual B-complexes*. The claim dual to that in Theorem 1 holds for these complexes.

Theorem 2. *The dual five-dimensional B-complexes C^5 are a manifold of two-dimensional planes intersecting some curve and tangent to tangentially nondegenerate hypersurfaces.*

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I. V. Bubyakin
North-Eastern Federal University, Yakutsk, Russia
`bubyakiniv@mail.ru`

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ALMOST CONTACT METRIC STRUCTURES DEFINED BY AN N–EXTENDED CONNECTION

S. V. Galaev

Abstract. On a manifold with an almost contact metric structure $(\varphi, \vec{\xi}, \eta, g, X, D)$, we introduce the notions of intrinsic and N-extended connections. Using an N-extended connection, we define a new contact metric structure on the distribution D, which is called an extended almost contact metric structure. The properties of this structure are studied.

Keywords: almost contact metric structure, intrinsic connection, N-extended connection, extended almost contact metric structure

1. Introduction

The study of the geometry of tangent bundles begins with the seminal work [1] of 1958 by Sasaki. Using a Riemannian metric g on a smooth manifold X, Sasaki defines a Riemannian metric G on the tangent bundle TX of X. Sasaki's construction is based on the natural splitting (holding due to the existence of the Levi-Civita connection on a Riemannian manifold) of the tangent bundle TTX of the manifold TX into a direct sum of a vertical distribution and a horizontal distribution whose fibers are isomorphic to the fibers of TX. An odd analog of the tangent bundle is given by the distribution D of an almost contact metric structure $(\varphi, \vec{\xi}, \eta, g)$. Because of the prescription of a connection over the distribution [2] (and then of an N-extended connection, namely, of a connection on the vector bundle (X, D)) the bundle TX splits just as the bundle TTX into the direct sum of a vertical distribution and a horizontal distribution. As was shown in [2, 3], D is thus endowed with a natural almost contact metric structure which makes it possible for example to give an invariant nature to the analytical description of mechanics with constraints. In [3], on the manifold D, the geodesic pulverization of a connection over a distribution is defined, which is an analog of the geodesic pulverization defined on the space of the tangent bundle TX, having the clear physical interpretation: the projections of the integral curves of the geodesic pulverization of a connection over a distribution coincide with the admissible geodesics (the trajectories of the motion of a mechanical system with constraints).

This article, intended an introduction to the geometry of extended almost contact metric structures, is devoted to the development of the two ideas: the idea of generalizing Sasaki's construction (see [1]) to the case of odd dimension as well as the idea of extending an intrinsic connection.

The article is organized as follows: Section 2 consists of three subsections, the first of which contains a summary of the intrinsic geometry of almost contact metric spaces. The reader is referred to [4] for a more detailed exposition.

In Subsection 2.2, we introduce the concept of N-extended metric connection. An intrinsic connection defines a parallel translation of the admissible vectors (i.e., the vectors belonging to D) along admissible curves. Each corresponding N-extending connection is a connection in the vector bundle (D, π, X) defined by the intrinsic connection and the endomorphism $N : D \to D$. The choice of the endomorphism $N : D \to D$ influences the properties of the extended connection as well as the properties of the (extended) almost contact metric structure appearing on the space D of the vector bundle (D, π, X). Central to this subsection is a theorem on the existence and uniqueness of an N-extended metric connection with zero torsion. In Section 2.3, we reveal the connection of the intrinsic and extended connections with the familiar connections arising on almost contact metric spaces.

In Section 3, we define an extended almost contact metric structure on a manifold D with extended metric connection. We study the properties of an extended almost contact metric structure. Particular attention is paid to almost contact Kähler spaces.

2. Intrinsic and N-Extended Connections

2.1. Main information from the intrinsic geometry of almost contact metric spaces. Let X be a smooth manifold of odd dimension n and let ΓTX be the $C^\infty(X)$-module of smooth vector fields on X. All manifolds, tensor fields, and other geometric objects are assumed smooth of class C^∞. An *almost contact metric structure* on X is a collection $(\varphi, \vec{\xi}, \eta, g)$ of tensor fields on X, where φ is a tensor of type (1, 1) which is called the *structure endomorphism*, $\vec{\xi}$ and η and a vector and a covector, called the *structure vector* and the *contact form*, g is a (pseudo-) Riemannian metric. Moreover,

$$\eta(\vec{\xi}) = 1, \quad \varphi(\vec{\xi}) = 0, \quad \eta \circ \varphi = 0, \quad \varphi^2 \vec{X} = -\vec{X} + \eta(\vec{X})\vec{\xi},$$

$$g(\varphi\vec{X}, \varphi\vec{Y}) = g(\vec{X}, \vec{Y}) - \eta(\vec{X})\eta(\vec{Y}), \quad d\eta(\vec{X}, \vec{\xi}) = 0,$$

$\vec{X}, \vec{Y} \in \Gamma TX$.

The skew-symmetric tensor $\Omega(\vec{X}, \vec{Y}) = g(\vec{X}, \varphi\vec{Y})$ is called the *fundamental form of the structure*. A manifold on which an almost contact metric structure is fixed is called an *almost contact metric manifold*. If $\Omega = d\eta$ then the almost contact metric structure is called a *contact metric structure*. An almost contact metric structure is called *normal* if $N_\varphi + 2d\eta \otimes \vec{\xi} = 0$, where N_φ is the Nijenhuis torsion generated by the tensor φ. A normal contact metric structure is called a *Sasakian structure*. A manifold with a Sasakian structure is called a *Sasakian manifold*. Let D be a smooth distribution of codimension 1 defined by a form η, and $D^\perp = \mathrm{Span}(\vec{\xi})$ is its rig. If the restriction of the form $\omega = d\eta$ to D is nondegenerate then $\vec{\xi}$ uniquely determined from the conditions $\eta(\vec{\xi}) = 1$, $\ker\omega = \mathrm{Span}(\vec{\xi})$, is called the *Reeb vector*.

Call an almost contact metric structure *almost normal* if

$$N_\varphi + 2(d\eta \circ \varphi) \otimes \vec{\xi} = 0. \tag{1}$$

In what follows, we refer to an almost normal almost contact metric space as an *almost contact Kähler space* if its fundamental form is closed. An almost metric space will be called an *almost K-contact metric space* if $L_{\vec{\xi}}g = 0$. The last equality is more frequently used in the case when the form ω has maximal rank; then the corresponding space is called *K-contact*.

An almost normal contact metric structure is obviously a Sasakian structure. Sasakian spaces are very popular among the researchers studying almost contact metric spaces for two main reasons. On the one hand, there are many interesting and informative examples of Sasakian structures; on the other hand, Sasakian manifolds possess very important and natural properties. At the same time, almost contact Kähler spaces inherit a number of important properties of Sasakian spaces, which turns out to rather substantial in the cases when the contact metric space cannot be a Sasakian space in principle [6].

Call a chart $K(x^\alpha)$ $(\alpha,\beta,\gamma=1,\dots,n)$ $(a,b,c,e=1,\dots,n-1)$ of a manifold X *adapted to a nonholonomic manifold* D if $D^\perp=\mathrm{Span}(\frac{\partial}{\partial x^n})$ [4]. Let $P:TX\to D$ be the projection defined by the decomposition $TX=D\oplus D^\perp$ and let $K(x^\alpha)$ be an adapted chart. The vector fields $P(\partial_a)=\vec{e}_a=\partial_a-\Gamma^n_a\partial_n$ are linearly independent and generate a distribution D: $D=\mathrm{Span}(\vec{e}_a)$, in the domain of the corresponding chart. Thus, on X, we get a nonholonomic field of the bases $(\vec{e}_a,\partial_n)$ and the corresponding field of the cobases $(dx^a,\theta^n=dx^n+\Gamma^n_a dx^a)$. It is a straightforward check that $[\vec{e}_a\vec{e}_b]=M^n_{ab}\partial_n$, where the components M^n_{ab} constitute the so-called *nonholonomy tensor* [7]. If we require that $\vec{\xi}=\partial_n$ in all adapted charts then, in particular, $[\vec{e}_a\vec{e}_b]=2\omega_{ba}\partial_n$, where $\omega=d\eta$. The basis $\vec{e}_a=\partial_a-\Gamma^n_a\partial_n$ will also be called adapted since it is defined by an adapted chart. Note that $\partial_n\Gamma^n_a=0$. Let $K(x^\alpha)$ and $K(x^{\alpha'})$ be adapted charts; then, under the condition $\vec{\xi}=\partial_n$, we obtain the following formulas for the change of coordinates: $x^\alpha=x^\alpha(x^{\alpha'})$, $x^n=x^{n'}+x^n(x^{\alpha'})$.

A tensor field of type (p,q) on an almost contact metric manifold will be called *admissible* (to the distribution D) if its coordinate representation in an adapted chart has the form

$$t=t^{a_1\dots a_p}_{b_1\dots b_q}\vec{e}_{a_1}\otimes\dots\otimes\vec{e}_{a_p}\otimes dx^{b_1}\otimes\dots\otimes dx^{b_q}.$$

The definition of almost contact structure implies that the affinor φ is an admissible tensor field of type $(1,1)$. Reckoning with the properties of the field of the affinor φ, we call it an *admissible almost complex structure*. It stands to reason to call the form $\omega=d\eta$, which is an admissible form too, an *admissible symplectic form*.

The transformation of the components of an admissible tensor field in adapted coordinates obeys the law

$$t^a_b=A^a_{a'}A^{b'}_b t^{a'}_{b'},$$

where $A^a_{a'}=\frac{\partial x^a}{\partial x^{a'}}$.

Remark 1. The formulas for the transformation of an admissible tensor field implies that the derivatives $\partial_n t^a_b$ are again the components of an admissible tensor field. Moreover, the vanishing of the derivatives $\partial_n t^a_b$ does not depend on the choice of adapted coordinates. This circumstance is supported by the fact that $(L_{\vec{\xi}}t)^a_b=\partial_n t^a_b$.

Remark 2. Refer to an admissible tensor structure for which $\partial_n t^a_b=0$ as projectable (other terms addressed to structures with such a property can be found in the literature: "basic," "semibasic," etc.). Admissible projectable structures can naturally be regarded as structures defined on a manifold of lesser dimension.

Using adapted coordinates, introduce the admissible tensor fields:

$$h^a_b=\frac{1}{2}\partial_n\varphi^a_b,\quad C_{ab}=\frac{1}{2}\partial_n g_{ab},\quad C^a_b=g^{da}C_{db},\quad \psi^b_a=g^{da}\omega_{da}.$$

We will use the notations for the connection and the coefficients of the Levi-Civita connection of the tensor g: $\widetilde{\nabla}$, $\widetilde{\Gamma}^\alpha_{\beta\gamma}$. Straightforward calculations justify the following

Theorem 1. *The coefficients of the Levi-Civita connection of an almost contact metric space in adapted coordinates have the form*

$$\widetilde{\Gamma}^c_{ab} = \Gamma^c_{ab}, \quad \widetilde{\Gamma}^n_{ab} = \omega_{ba} - C_{ab}, \quad \widetilde{\Gamma}^b_{an} = \widetilde{\Gamma}^b_{na} = C^b_a - \psi^b_a, \quad \widetilde{\Gamma}^n_{na} = 0, \quad \widetilde{\Gamma}^a_{nn} = 0,$$

where

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\vec{e}_b g_{cd} + \vec{e}_c g_{bd} - \vec{e}_d g_{bc}).$$

2.2. An N-extended metric connection. Under an *intrinsic linear connection* on a manifold with almost contact metric structure [4] we mean a mapping $\nabla : \Gamma D \times \Gamma D \to \Gamma D$ satisfying the conditions:

(1) $\nabla_{f_1 \vec{u_1} + f_2 \vec{u_2}} = f_1 \nabla_{\vec{u_1}} + f_2 \nabla_{\vec{u_2}}$,

(2) $\nabla_{\vec{u}} f \vec{v} = f \nabla_{\vec{u}} \vec{v} + (\vec{u} f)\vec{v}$,

where ΓD is the module of admissible vector fields. The coefficients of the linear connection are defined from the relation $\nabla_{\vec{e}_a} \vec{e}_b = \Gamma^c_{ab} \vec{e}_c$.

The torsion of the intrinsic linear connection S is by definition

$$S(\vec{X}, \vec{Y}) = \nabla_{\vec{X}} \vec{Y} - \nabla_{\vec{Y}} \vec{X} - P[\vec{X}, \vec{Y}].$$

Thus, in adapted coordinates, we have $S^c_{ab} = \Gamma^c_{ab} - \Gamma^c_{ba}$.

The action of an intrinsic linear connection naturally extends to arbitrary admissible tensor fields. An important example of an intrinsic linear connection is given by the intrinsic metric connection determined uniquely by the conditions $\nabla g = 0$, $S = 0$ [7]. In adapted coordinates, we have

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\vec{e}_b g_{cd} + \vec{e}_c g_{bd} - \vec{e}_d g_{bc}).$$

Note that $\Gamma^a_{bc} = \widetilde{\Gamma}^a_{bc}$ (see Theorem 1).

Like a connection in the ambient space, an intrinsic linear connection can be given by defining a horizontal distribution over the space of a vector bundle. In the case of an intrinsic connection, as such a bundle, there acts the distribution D. We say that *a connection is defined* over a distribution D if the distribution $\widetilde{D} = \pi_*^{-1}(D)$, where $\pi : D \to X$, splits into the direct sum of the form $\widetilde{D} = HD \oplus VD$, where VD is the vertical distribution on the total space D.

Endow D with the structure of a smooth manifold by assigning to each adapted chart $K(x^\alpha)$ on X the superchart $\tilde{K}(x^\alpha, x^n + a)$ on the manifold D, where x^{n+a} are the coordinates of an admissible vector in the basis $\vec{e}_a = \partial_a - \Gamma^n_a \partial_n$. The so-constructed superchart will also be called *adapted*. The definition of connection over a distribution is equivalent to the definition of $G^a_b(x^\alpha, x^{n+a})$ such that $HD = \operatorname{Span}(\vec{\varepsilon}_a)$, where $\vec{\varepsilon}_a = \partial_a - \Gamma^n_a \partial_n - G^b_a \partial_{n+b}$. If $G^a_b(x^a, x^{n+a}) = \Gamma^a_{bc}(x^a) x^{n+c}$ then the connection over the distribution is defined by an intrinsic linear connection. The notion of extended connection was introduced in [2]. An extended connection is always considered with respect to some connection over a distribution and is defined by the expansion $TD = \widetilde{HD} \oplus VD$, where $HD \subset \widetilde{HD}$. The extended connection is a connection in a vector bundle. As follows from the definition of extended connection, from its definition (under the condition of an already existing connection over a distribution), it suffices to define a vector field $\vec{u}$ on the manifold D having the following coordinate representation: $\vec{u} = \partial_n - N^a_b x^{n+b} \partial_{n+a}$, where the endomorphism $N : D \to D$ can be chosen arbitrarily. We refer as the *torsion of an extended connection* to the torsion of the initial intrinsic connection. In what follows, we refer to an extended connection as an *N-extended connection.*

In [7], Vagner refers to the admissible tensor field defined by the equality

$$R(\vec{u},\vec{v})\vec{w} = \nabla_{\vec{u}}\nabla_{\vec{u}}\vec{w} - \nabla_{\vec{v}}\nabla_{\vec{u}}\vec{w} - \nabla_{p[\vec{u},\vec{v}]}\vec{w} - p[q[\vec{u},\vec{v}]\vec{w}]$$

as the *first Schouten curvature tensor.* The coordinate representation of the Schouten tensor in adapted coordinates has the form

$$R^d_{abc} = 2\vec{e}[a\Gamma^d_{b]c} + 2\Gamma^d_{a|e|}\Gamma^e_{b]c}.$$

If the distribution D does not contain an integrable distribution of dimension $n-2$, the vanishing of the Schouten curvature tensor is equivalent to the fact that the parallel translation of admissible vectors along admissible curves does not depend on the translation path [7]. Call the Schouten tensor the *curvature tensor of the distribution* D, and if the Schouten tensor vanishes then call D a *distribution of zero curvature*. It is not hard to establish that the partial derivatives $\partial_n\Gamma^a_{bc} = P^a_{bc}$ are the components of an admissible vector field [7].

Remark 3. For (almost) K-contact spaces, the Schouten curvature tensor possesses the same properties as the curvature tensor of a Riemannian manifold. It is not so in the general case.

The vector fields $\big(\vec{\varepsilon}_a = \partial_a - \Gamma^n_a\partial_n - G^b_{ac}x^{n+c}\partial_{n+b}, \vec{u} = \partial_n - G^a_n\partial_{n+a}, \partial_{n+a}\big)$ define a nonholonomic (adapted) basis field D, and the forms $\big(dx^a, \theta^n = dx^n + \Gamma^n_a dx^a, \theta^{n+a} = dx^{n+a} + \Gamma^a_{bc}dx^b + N^a_b x^{n+b}dx^n\big)$ determine the corresponding cobasis field. Carrying out the necessary calculations, we obtain the following structure equations:

$$[\vec{\varepsilon}_a,\vec{\varepsilon}_b] = 2\omega_{ba}\vec{u} + x^{n+d}\big(2\omega_{ba}N^c_d + R^c_{bad}\big)\partial_{n+c}, \tag{3}$$

$$[\vec{\varepsilon}_a,\vec{u}] = x^{n+d}\big(\partial_n\Gamma^c_{ad} - \nabla_a N^c_d\big)\partial_{n+c}, \tag{4}$$

$$[\vec{\varepsilon}_a,\partial_{n+b}] = \Gamma^c_{ab}\partial_{n+c}.$$

From (3) and (4) we get the expression for the curvature tensor of the extended connection:

$$K(\vec{u},\vec{v})\vec{w} = 2\omega(\vec{u},\vec{v})N\vec{w} + R(\vec{u},\vec{v})\vec{w},$$

$$K(\vec{\xi},\vec{u})\vec{v} = P(\vec{u},\vec{v}) - (\nabla_{\vec{u}}N)\vec{v},$$

where $\vec{u},\vec{v}\in\Gamma D$.

Theorem 2. *There exists an N-extended metric connection defined uniquely by the following conditions:*

(1) $\vec{Z}g(\vec{X},\vec{Y}) = g(\nabla_{\vec{Z}}\vec{X},\vec{Y}) + g(\vec{X},\nabla_{\vec{Z}}\vec{Y})$ *(the metric property),*

(2) $\nabla_{\vec{X}}\vec{Y} - \nabla_{\vec{Y}}\vec{X} - p[\vec{X},\vec{Y}] = 0$ *(the absence of torsion),*

(3) *N is a symmetric operator such that*

$$g(N\vec{X},\vec{Y}) = \frac{1}{2}L_{\vec{\xi}}g(\vec{X},\vec{Y}), \tag{5}$$

where $\vec{X},\vec{Y},\vec{Z}\in\Gamma D$ are sections of D, and $P: TX\to D$ is the projection.

Proof. The first two conditions of the theorem uniquely define the intrinsic metric connection [7]. Alternating the second covariant derivative, we get $\nabla_{[e}\nabla_{a]}g_{bc} = 2\omega_{ea}\partial_n g_{bc} - g_{dc}R^d_{eab} - g_{bd}R^d_{eac}$.

Comparing the above result with (5), we find an explicit expression for the endomorphism N:

$$N^f_b = \frac{1}{4(n-1)}\omega^{ea}(R^f_{eab} + g_{bd}g^{cf}R^d_{eac}).$$

If $\partial_n g_{ab} = 0$ then put $N = 0$. The theorem is proved.

Refer to an N-extended connection furnished with the properties of Theorem 2 as an *N-extended metric connection.* For an extended connection, use the notation $\nabla^N = (\nabla, N)$; in the particular case, $\nabla^1 = (\nabla, 0)$.

3. Special Connections of Manifolds with Almost Contact Metric Structure

Cartan (see [8–10]) was the first to consider a linear metric connection with torsion instead of the Levi-Civita connection. Most interesting among metric connections with torsion is the semisymmetric connection treated systematically by Yano in [11]. The quarter-symmetric connection was defined in 1975 by Golab in [12]. Many works are devoted both to metric and nonmetric connections with torsion defined on manifolds with almost contact metric structure. Here we will only dwell on Bejancu's article [13]. Bejancu defines the connection ∇^B on a Sasakian manifold by the formula

$$\nabla^B_{\vec{X}} = \widetilde{\nabla}_{\vec{X}}\vec{Y} - \eta(\vec{X})\widetilde{\nabla}_{\vec{Y}}\vec{\xi} - \eta(\vec{Y})\widetilde{\nabla}_{\vec{X}}\vec{\xi} + (\omega + c)(\vec{X}, \vec{Y})\vec{\xi}.$$

In adapted coordinates, the nonzero components $\Gamma^{B\alpha}_{\beta\gamma}$ of the connection ∇^B are

$$\Gamma^{Ba}_{bc} = \Gamma^a_{bc} = \frac{1}{2}g^{ad}(\vec{e}_b g_{cd} + \vec{e}_c g_{bd} - \vec{e}_d g_{bc}).$$

The connection constructed by Bejancu is in general not metric in the more general case of an almost contact metric structure than the Sasakian structure. Indeed, since $\nabla^B_n g_{ab} = \partial_n g_{ab}$, the metricity of the Bejancu connection is equivalent to the almost K-contactness of the contact metric structure. On a manifold with an almost contact metric structure, define the connection ∇^N by the equality

$$\nabla^N_{\vec{X}} = \nabla^B_{\vec{X}}\vec{Y} + \eta(\vec{X})N\vec{Y},$$

where N is the endomorphism of Theorem 2. Refer to the so-introduced connection as the *N-connection.* The nonzero components of the N-connection are at most

$$\Gamma^{Na}_{bc} = \Gamma^a_{bc} = \frac{1}{2}g^{ad}(\vec{e}_b g_{cd} + \vec{e}_c g_{bd} - \vec{e}_d g_{bc}),$$

$\Gamma^{Na}_{nc} = N^a_c$. The curvature of the N-connection is defined by the equality

$$S^N(\vec{X}, \vec{Y}) = 2\omega(\vec{X}, \vec{Y})\vec{\xi} + \eta(\vec{X})N\vec{Y} - \eta(\vec{Y})N\vec{X}.$$

Straightforward calculations in adapted coordinates validate the following

Theorem 3. *The N-connection is a metric connection.*

4. An N-Extended Connection as an Almost Contact Metric Structure

Suppose that a manifold X is endowed with a contact metric structure $(D, \varphi, \vec{\xi}, \eta, g, X)$. On the distribution D as a smooth manifold, define an almost contact metric structure $(\tilde{D}, J, \vec{u}, \lambda = \eta \circ \pi_*, \tilde{g}, D)$ by setting

$$\tilde{g}(\vec{\varepsilon}_a, \vec{\varepsilon}_b) = \tilde{g}(\partial_{n+a}, \partial_{n+b}) = g(\vec{e}_a, \vec{e}_b), \quad \tilde{g}(\vec{\varepsilon}_a, \partial_{n+b}) = \tilde{g}(\vec{\varepsilon}_a, \vec{u}) = \tilde{g}(\vec{u}, \partial_{n+b}) = 0,$$

$$J(\vec{\varepsilon}_a) = \partial_{n+a}, \quad J(\partial_{n+a}) = -\vec{\varepsilon}_a.$$

Here the vector fields $(\vec{\varepsilon}_a = \partial_a - \Gamma^n_a \partial_n - G^b_{ac} x^{n+c} \partial_{n+b}, \vec{u} = \partial_n - G^a_n \partial_{n+a}, \partial_{n+a})$ are defined by the extended connection. The so-obtained structure will be called an *extended almost contact metric structure*. Let $\tilde{\omega} = d\lambda$. It is a straightforward check that the nonzero components of the form $\tilde{\omega}$ are defined by the equalities $\tilde{\omega}_{ab} = \omega_{ab}$. Thus, $\operatorname{rk}\tilde{\omega} = \frac{n-1}{2}$. This in particular implies that the constructed structure is not contact and in particular not a Sasakian structure.

Theorem 4. *An extended almost contact metric structure is almost K-contact if and only if the initial structure is K-contact.*

Proof. The nonzero components of the Lie derivative $L_{\vec{u}}\tilde{g}$ have the following form in adapted coordinates:

$$(L_{\vec{u}}\tilde{g})_{ab} = \partial_n g_{ab}, \tag{6}$$

$$(L_{\vec{u}}\tilde{g})_{n+a,n+b} = \partial_n g_{ab} - g_{ac} N^c_b - -g_{cb} N^c_a, \tag{7}$$

$$(L_{\vec{u}}\tilde{g})_{n+a,b} = g_{ac}(P^c_{bd} - \nabla_b N^c_d) x^{n+d}. \tag{8}$$

The components in (7) are also zero as the components of the covariant derivatives of the metric tensor. The equality $\partial_n g_{ab} = 0$ implies the two other equalities: $N^c_d = 0$ and $P^c_{bd} = 0$ (see (6) and (8)), which proves the theorem.

Assume that the initial structure is K-contact ($N = 0$). Then we have

Theorem 5. *An almost contact metric structure $(\tilde{D}, J, \vec{u}, \lambda = \eta \circ \pi_*, \tilde{g}, D)$ is almost normal if and only if D is a distribution of zero curvature.*

Proof. Rewrite (1) in new notations:

$$N_J + 2(d\tilde{\eta} \circ J) \circ \vec{u} = 0.$$

It was proved in [4] that an almost contact structure is almost normal if and only if $\tilde{P} \circ N_J = 0$, where $\widetilde{P} : TD \to \widetilde{D}$ is the projection. Using (3)–(5) for the connection ∇^1, we obtain the following two components of the Nijenhuis affinor J:

$$N_J(\vec{\varepsilon}_a, \vec{\varepsilon}_b) = -R^e_{abc} x^{n+c} \partial_{n+e},$$

$$N_J(\partial_{n+a}, \partial_{n+b}) = 2\omega_{ba} + R^e_{abc} x^{n+c} \partial_{n+e},$$

$$N_J(\vec{\varepsilon}_a, \partial_{n+b}) = 0,$$

$$N_J(\vec{\varepsilon}_a, \partial_n) = N_J(\partial_{n+a}, \partial_n) = -x^{n+c} P^b_{ac} \partial_{n+b}.$$

Thus, an extended almost contact metric structure is almost normal if and only if the Schouten curvature tensor is zero.

Theorem 6. *An almost contact metric structure $(\tilde{D}, J, \vec{u}, \lambda = \eta \circ \pi_*, \tilde{g}, D)$ is an almost contact Kähler structure if and only if $(\varphi, \vec{\xi}, \eta, g)$ is a Sasakian structure with zero curvature distribution.*

Proof. Straightforward calculations yield $d\Omega = 0 \Leftrightarrow d\tilde{\Omega} = 0$, where $\tilde{\Omega}(\vec{X}, \vec{Y}) = \tilde{g}(\vec{X}, J\vec{Y})$, which proves the theorem.

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S. V. Galaev
Chernyshevskiĭ Saratov State University, Saratov
`sgalaev@mail.ru`

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A NONLOCAL PROBLEM ON THE SEMIAXIS FOR DEGENERATE EVOLUTION EQUATIONS

N. D. Ivanova and V. E. Fedorov

Abstract. We obtain necessary and sufficient conditions of the unique solvability in the classical and generalized sense of a time nonlocal boundary value problem with an integral condition on the semiaxis for a linear homogeneous differential equation of the first order in Banach space with a degenerate operator at the derivative. The conditions on the operators in this equation ensure the exponential decay of the respective strongly continuous resolving semigroup. An estimate exhibiting the exponential decay of a generalized solution is given. The abstract results are used to examine a time nonlocal boundary value problem for a class of partial differential equations with polynomials of the Laplacian, including some equations of filtration theory and the theory of semiconductors.

Keywords: nonlocal problem, degenerate evolution equation, operator semigroup, classical solution, generalized solution, boundary value problem

1. Introduction

Consider the nonlocal problem

$$\int_0^\infty u(t)\eta(t)\,dt = u_0, \tag{1.1}$$

$$L\dot{u}(t) = Mu(t), \quad t \geq 0, \tag{1.2}$$

where the evolution equation is degenerate, $L \in \mathscr{L}(\mathfrak{U};\mathfrak{V})$ is a linear operator from a Banach space $\mathfrak{U}$ in a Banach space $\mathfrak{V}$, $\ker L \neq \{0\}$, $M \in \mathscr{C}l(\mathfrak{U};\mathfrak{V})$ is a linear closed operator with domain $D(M)$ dense in $\mathfrak{U}$, acting in $\mathfrak{V}$, and $\eta : (0,\infty) \to \mathbb{R}$ is a nonnegative nonincreasing function. The condition of the strong (L,p)-radiality of M [1] is assumed, which ensures existence of a strongly continuous degenerate resolving semigroup for (1.2) that decays exponentially.

The problems for equations of this form with a degenerate operator at the derivative and thus unsolvable with respect to it are an abstract form of boundary value problems for partial differential equations and systems of these equations convenient for the study by operator methods [1–4]. In particular, the necessary and sufficient conditions of solvability of (1.1), (1.2) obtained in the present article are used to establish unique solvability of boundary value problems for (1.2) with L and M polynomials of an elliptic operator in the space variables. This class includes some equations of filtration theory and the theory of semiconductors.

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This article is a continuation of [5] to the case of degenerate evolution equation (1.2). The article contains necessary and sufficient conditions of generalized and classical solvability of (1.1) with a nonnegative nonincreasing function η for the equation

$$\dot{u}(t) = Au(t), \quad t \geq 0, \tag{1.3}$$

solvable with respect to the derivative, where A is a linear operator generating a strongly continuous semigroup of class C_0 [6] in a Banach space $\mathfrak{X}$. At the same time the results were complemented in [7], where the degenerate equation (1.2) is examined with the nonlocal boundary condition

$$\int_0^T u(t)\eta(t)\, dt = u_0 \tag{1.4}$$

and the unique solvability of this problem is studied for the nonhomogeneous equations (1.2) and (1.3). It is observed in [7] that in contrast to the homogeneous equation the behavior at infinity of a solution to the nonhomogeneous equation is not defined by the properties of A, L, or M; hence, the well-posedness of the nonlocal condition (1.1) is not ensured in terms of these operators. This is the reason why we consider only the homogeneous equation.

Note also the articles by I. V. Tikhonov devoted to uniqueness of solutions to (1.2), (1.4) and (1.3), (1.4) [8] and unique solvability of (1.1), (1.2) and (1.3), (1.4) [9] and the articles by A. A. Kerefov, [10], V. V. Shelukhin [11], A. I. Kozhanov [12], and many other authors (see, for instance, [13–15]), where close nonlocal problems are studied. Somewhat more detailed historiography can be found in [7].

2. Preliminaries

Given a function $\eta : (0, +\infty) \to \mathbb{R}$, consider the nonlocal problem

$$\int_0^\infty x(t)\eta(t)\, dt = x_0, \tag{2.1}$$

$$\dot{x}(t) = Ax(t), \quad t \geq 0, \tag{2.2}$$

where A is a closed linear operator with dense domain $D(A)$ in a Banach space $\mathfrak{X}$ which generates a strongly continuous semigroup $\{X(t) \in \mathscr{L}(\mathfrak{X}) : t \geq 0\}$ of class C_0.

As in [5], a function $x(t) = X(t)v$, $t \geq 0$, $v \in \mathfrak{X}$ is a *generalized solution* to (2.2). In our case this function is continuous but possibly not differentiable.

A function $x \in C^1([0,\infty); \mathfrak{X})$ is referred to as a *classical solution* to (2.2) if (2.2) holds for every $t \geq 0$. Clearly, for given an operator A, every classical solution to (2.2) is a generalized solution and a generalized solution is classical whenever $v \in D(A)$.

A generalized or classical solution to (2.2) satisfying (2.1) is called a *generalized* or *classical solution to* (2.1), (2.2).

Theorem 2.1 [5]. *Assume that an operator A is a generator of a strongly continuous semigroup $\{X(t) \in \mathscr{L}(\mathfrak{X}) : t \geq 0\}$ of class C_0, the inequality $\|X(t)\|_{\mathscr{L}(\mathfrak{X})} \leq Ke^{-\alpha t}$ holds for $t \geq 0$ with constants $K > 0$ and $\alpha > 0$, the function η is nonnegative and nonincreasing on $(0,\infty)$ and $\eta(t) > 0$ as $t \to 0+$. Then*

(i) *for every $x_0 \in D(A)$, there exists a unique generalized solution $x(t) = X(t)v$ to* (2.1), (2.2) *such that $\|x(t)\|_{\mathfrak{X}} \leq Ce^{-\alpha t}\|Ax_0\|_{\mathfrak{X}}$, where the constant C is independent of x_0 and t;*

(ii) *a generalized solution to* (2.1), (2.2) *is classical if and only if $x_0 \in D(A^2)$;*

(iii) *if $x_0 \in \mathfrak{X} \setminus D(A)$ then a generalized solution to* (2.1), (2.2) *does not exist.*

Proceed with the degenerate evolution equation

$$L\dot{u}(t) = Mu(t), \quad t \geq 0. \tag{2.3}$$

Here $\mathfrak{U}$ and $\mathfrak{V}$ are Banach spaces, $L \in \mathscr{L}(\mathfrak{U};\mathfrak{V})$, $\ker L \neq \{0\}$, and $M \in \mathscr{C}l(\mathfrak{U};\mathfrak{V})$. Put $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathscr{L}(\mathfrak{V};\mathfrak{U})\}$, $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$, $R^L_\mu(M) = (\mu L - M)^{-1}L$, and $L^L_\mu = L(\mu L - M)^{-1}$.

Let $p \in \mathbb{N}_0$. An operator M is *strongly* (L,p)*-radial* with constants $K > 0$ and $a \in \mathbb{R}$ if

(i) $\exists a \in \mathbb{R}$ $(a,\infty) \subset \rho^L(M)$;

(ii) $\exists K > 0$ $\forall \mu \in (a,\infty)$ $\forall n \in \mathbb{N}$

$$\max\left\{\left\|\left(R^L_\mu(M)\right)^{n(p+1)}\right\|_{\mathscr{L}(\mathfrak{U})}, \left\|\left(L^L_\mu(M)\right)^{n(p+1)}\right\|_{\mathscr{L}(\mathfrak{V})}\right\} \leq \frac{K}{(\mu - a)^{n(p+1)}};$$

(iii) there exists a subspace $\overset{\circ}{\mathfrak{V}}$ dense in $\mathfrak{V}$ and such that

$$\left\|M(\mu L - M)^{-1}\left(L^L_\mu(M)\right)^{p+1} f\right\|_{\mathfrak{V}} \leq \frac{\mathrm{const}(f)}{(\mu - a)^{p+2}} \quad \forall f \in \overset{\circ}{\mathfrak{V}},$$

$$\left\|\left(R^L_\mu(M)\right)^{p+1}(\mu L - M)^{-1}\right\|_{\mathscr{L}(\mathfrak{V};\mathfrak{U})} \leq \frac{K}{(\mu - a)^{p+2}}$$

for every $\mu \in (a,\infty)$.

Remark 2.1. The equivalence of the conditions of this definition, similar to more cumbersome conditions in [1], is demonstrated in [16].

Some family of operators $\{U(t) \in \mathscr{L}(\mathfrak{U}) : t \geq 0\}$ is the *resolving semigroup* of (2.3) if

(i) $U(s)U(t) = U(s+t)$, $s,t \geq 0$;

(ii) for every u_0 from some dense subspace in $\mathfrak{U}$, the function $u(t) = U(t)u_0$ is a classical solution to (2.3);

(iii) $\operatorname{im} V(0) \subset \operatorname{im} U(0)$ for every operator family $\{V(t) \in \mathscr{L}(\mathfrak{U}) : t \geq 0\}$ satisfying (i) and (ii).

Put $\mathfrak{U}^0 = \ker(R^L_\mu(M))^{p+1}$ and $\mathfrak{V}^0 = \ker(L^L_\mu(M))^{p+1}$. The symbols $\mathfrak{U}^1$ and $\mathfrak{V}^1$ stand for the closure of the images of $\operatorname{im}(R^L_\mu(M))^{p+1}$ and $\operatorname{im}(L^L_\mu(M))^{p+1}$ in $\mathfrak{U}$ and $\mathfrak{V}$, respectively.

Denote the restriction of L (M) to $\mathfrak{U}^k$ $(D(M_k) = D(M) \cap \mathfrak{U}^k)$, $k = 0,1$, by L_k (M_k).

Theorem 2.2 [1]. *Let an operator M be strongly (L,p)-radial with constants $K > 0$ and $a \in \mathbb{R}$. Then*

(i) $\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1$ *and* $\mathfrak{V} = \mathfrak{V}^0 \oplus \mathfrak{V}^1$;

(ii) $L_k \in \mathscr{L}(\mathfrak{U}^k;\mathfrak{V}^k)$ *and* $M_k \in \mathscr{C}l(\mathfrak{U}^k;\mathfrak{V}^k)$, $k = 0,1$;

(iii) *there exist* $M_0^{-1} \in \mathscr{L}(\mathfrak{V}^0;\mathfrak{U}^0)$ *and* $L_1^{-1} \in \mathscr{L}(\mathfrak{V}^1;\mathfrak{U}^1)$;

(iv) $H = M_0^{-1}L_0$ *is nilpotent of degree at most* p;

(v) *there exists a strongly continuous semigroup* $\{U(t) \in \mathscr{L}(\mathfrak{U}) : t \geq 0\}$ *resolving* (2.3) *such that* $\|U(t)\|_{\mathscr{L}(\mathfrak{U})} \leq Ke^{at}$ *for all* $t \geq 0$;

(vi) $S = L_1^{-1}M_1$ *is a generator of a* C_0*-continuous semigroup* $\{U_1(t) \equiv U(t)|_{\mathfrak{U}^1} \in \mathscr{L}(\mathfrak{U}^1) : t \geq 0\}$ *of operators.*

Remark 2.2. In the case of $\ker L \neq \{0\}$ in Theorem 2.2, the unity $U(0)$ of the resolving semigroup is a nontrivial projection such that $\ker L \subset \ker U(0) = \mathfrak{U}^0$ and $\operatorname{im} U(0) = \mathfrak{U}^1$.

3. A Nonlocal Problem for a Degenerate Evolution Equation

Consider the nonlocal problem

$$\int_0^\infty u(t)\eta(t)\,dt = u_0, \tag{3.1}$$

$$L\dot{u}(t) = Mu(t), \quad t \geq 0, \tag{3.2}$$

for a homogeneous degenerate evolution equation assuming that M is strongly (L,p)-radial. By Theorem 2.2, problem (3.1), (3.2) is reduced to the two problems

$$\int_0^\infty x(t)\eta(t)\,dt = U(0)u_0, \tag{3.3}$$

$$\dot{x}(t) = Sx(t), \quad t \geq 0, \tag{3.4}$$

and

$$\int_0^\infty y(t)\eta(t)\,dt = (I - U(0))u_0, \tag{3.5}$$

$$H\dot{y}(t) = y(t), \quad t \geq 0, \tag{3.6}$$

where $x(t) \equiv U(0)u(t)$, $y(t) = (I - U(0))u(t)$ for $t \geq 0$, $S = L_1^{-1}M_1 \in \mathscr{C}l(\mathfrak{U}^1)$, and $H = M_0^{-1}L_0 \in \mathscr{L}(\mathfrak{U}^0)$.

Since H is nilpotent (item (iv) of Theorem 2.2), (3.6) has only trivial solution $y(t) \equiv 0$ (see, for instance, [1]). Hence, (3.5), (3.6) is solvable if and only if $(I - U(0))u_0 = 0$. Therefore, for $u_0 \in \mathfrak{U}^1$, problem (3.1), (3.2) is equivalent to (3.3), (3.4).

A function $u(t) = U(t)v$ for $v \in \mathfrak{U}$ is a *generalized solution* to (3.2) A function $u \in C^1([0,\infty);\mathfrak{U})$ is a *classical solution* to (3.2) if (3.2) is fulfilled in the classical sense. Every classical solution u to (3.2) is a classical solution to (3.4) in view of the above arguments and so $u(t) = U_1(t)v_1 = U(t)v_1 = U(t)(v_0 + v_1) = U(t)v$, where $v_0 \in \mathfrak{U}^0$, $v_1 \in \mathfrak{U}^1$, $v = v_0 + v_1$; as a consequence, it is also a generalized solution to (3.2). We involve the fact that $U(t)v_0 = U(t)U(0)v_0 = 0$ for every $v_0 \in \mathfrak{U}^0$ by Remark 2.2.

The same reasoning and the equality $D(S) = D(M_1)$ valid due to the continuous invertibility of L_1 imply that a generalized solution $u(t) = U(t)v$ to the equation (3.2) is classical whenever $v \in \mathfrak{U}^0 \dot{+} D(M_1)$.

A generalized or classical solution to (3.2) satisfying (3.1) is a *generalized* or *classical solution* to (3.1), (3.2).

Theorem 3.1. *Assume that M is strongly (L,p)-radial with constants $K > 0$ and $a < 0$, while $\eta : (0,\infty) \to \mathbb{R}$ is nonnegative and nonincreasing and does not vanish identically. Then*

(i) *there exists a unique generalized solution $u \in C([0,+\infty);\mathfrak{U})$ to* (3.1), (3.2) *for $u_0 \in D(M_1)$; in this event $\|u(t)\|_{\mathfrak{U}} \leq Ce^{-|a|t}\|Mu_0\|_{\mathfrak{V}}$ for all $t \geq 0$, where the constant C is independent of u_0 and t;*

(ii) *if $u_0 \in \mathfrak{U} \setminus D(M_1)$ then a generalized solution to* (3.1), (3.2) *does not exist;*

(iii) *a generalized solution to problem* (3.1), (3.2) *is classical if and only if $u_0 \in D\big((L_1^{-1}M_1)^2\big)$.*

Proof. As was noted, the condition $u_0 \in \mathfrak{U}^1$ is necessary for generalized solvability of (3.1), (3.2), in this case (3.1), (3.2) and (3.3), (3.4) are equivalent. By Theorem 2.2, the operator $S = L_1^{-1}M_1 \in \mathscr{C}l(\mathfrak{U}^1)$ is a generator of the semigroup $\{U_1(t) \in \mathscr{L}(\mathfrak{U}^1) : t \geq 0\}$ satisfying

$$\|U_1(t)\|_{\mathscr{L}(\mathfrak{U}^1)} \leq \|U(t)\|_{\mathscr{L}(\mathfrak{U})} \leq Ke^{at} = Ke^{-|a|t},$$

since a is negative by the conditions of the theorem. Applying Theorem 2.1 to (3.3), (3.4), we obtain the claim taking it into account that

$$\|u(t)\|_{\mathfrak{U}} = \|u(t)\|_{\mathfrak{U}^1} \leq C_1 e^{-|a|t}\|Su_0\|_{\mathfrak{U}^1}$$
$$\leq C_1\big\|L_1^{-1}\big\|_{\mathscr{L}(\mathfrak{V}^1;\mathfrak{U}^1)} e^{-|a|t}\|M_1u_0\|_{\mathfrak{V}^1} = Ce^{-|a|t}\|Mu_0\|_{\mathfrak{V}}. \quad \square$$

4. A Time Nonlocal Boundary Value Problem for a Certain Class of Partial Differential Equations

Let the polynomials $P_n(\lambda) = \sum_{i=0}^{n} c_i\lambda^i$ and $Q_m(\lambda) = \sum_{j=0}^{m} d_j\lambda^j$, with $c_i, d_j \in \mathbb{C}$, $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, m$, be such that $c_n, d_m \neq 0$, $m \geq n$. Next, $\Omega \subset \mathbb{R}^s$ is a bounded domain with boundary $\partial\Omega$ of class C^∞, $\eta : [0, \infty) \to \mathbb{R}$, Δ is the Laplace operator, and $\theta \in \mathbb{R}$. Consider the boundary value problem

$$\int_0^\infty z(x,t)\eta(t)\,dt = z_0(x), \quad x \in \Omega, \tag{4.1}$$

$$P_n(\Delta)\frac{\partial z}{\partial t}(x,t) = Q_m(\Delta)z(x,t), \quad (x,t) \in \Omega \times [0,\infty), \tag{4.2}$$

$$\begin{gathered} \theta\frac{\partial}{\partial n}\Delta^k z(x,t) + (1-\theta)\Delta^k z(x,t) = 0, \\ k = 0,\ldots,m-1,\ (x,t) \in \partial\Omega \times [0,\infty). \end{gathered} \tag{4.3}$$

Put

$$z(\cdot,t) = u(t), \quad t \geq 0,$$

$$\mathfrak{U} = H_\theta^{2n}(\Omega)$$
$$= \left\{v \in H^{2n}(\Omega) : \theta\frac{\partial}{\partial n}\Delta^k v(x) + (1-\theta)\Delta^k v(x) = 0,\ k = 0,\ldots,n-1,\ x \in \partial\Omega\right\},$$

$$\mathfrak{V} = L_2(\Omega), \quad L = P_n(\Delta), \quad M = Q_m(\Delta),$$

$$D(M) = H_\theta^{2m}(\Omega)$$
$$= \left\{v \in H^{2m}(\Omega) : \theta\frac{\partial}{\partial n}\Delta^k v(x) + (1-\theta)\Delta^k v(x) = 0,\ k = 0,\ldots,m-1,\ x \in \partial\Omega\right\}.$$

Thus, we reduced (4.1)–(4.3) to (3.1), (3.2).

Denote by λ_k, $k \in \mathbb{N}$ the eigenvalues of the operator A_1 enumerated with multiplicity counted which acts in $L_2(\Omega)$ and $A_1u = \Delta u$ on its domain

$$H_\theta^2(\Omega) = \left\{v \in H^2(\Omega) : \theta\frac{\partial}{\partial n}\Delta v(x) + (1-\theta)\Delta v(x) = 0,\ x \in \partial\Omega\right\}$$

Moreover, we assume that $\{\varphi_k : k \in \mathbb{N}\}$ is the corresponding system of eigenfunctions of this operator orthonormal in $L_2(\Omega)$ with the inner product $\langle \cdot, \cdot \rangle_{L_2(\Omega)}$. It is possible that $P_n(\lambda_k) = 0$ for some $k \in \mathbb{N}$, i.e., (4.2) is not solvable with respect to the time derivative z_t.

All functions of the form

$$z(x,t) = \sum_{P_n(\lambda_k)\neq 0} e^{t\frac{Q_m(\lambda_k)}{P_n(\lambda_k)}} \langle v, \varphi_k \rangle_{L_2(\Omega)} \varphi_k(x), \quad v \in H_\theta^{2n}(\Omega)$$

are generalized solutions to (4.1)–(4.3).

Theorem 4.1. *Assume that* $m > n$, $(-1)^{m-n} \operatorname{Re}(d_m/c_n) \le 0$, *the spectrum* $\sigma(A_1)$ *does not contain common roots of the polynomials* P_n *and* Q_m,

$$a = \sup_{P_n(\lambda_k)\neq 0} \operatorname{Re} \frac{Q_m(\lambda_k)}{P_n(\lambda_k)} < 0,$$

and $\eta : (0,\infty) \to \mathbb{R}$ *is nonnegative and nonincreasing and does not vanish identically. Then, for every* $z_0 \in H_\theta^{2m}(\Omega) \cap \operatorname{span}\{\varphi_k : P_n(\lambda_k) \neq 0\}$, *there exists a unique generalized solution to* (4.1)–(4.3) *and*

$$\exists C > 0 \ \forall t \ge 0 \quad \|z(\cdot,t)\|_{H^{2n}(\Omega)} \le Ce^{-|a|t}\|z_0\|_{H^{2m}(\Omega)}.$$

If

$$z_0 \notin H_\theta^{2m}(\Omega) \cap \operatorname{span}\{\varphi_k : P_n(\lambda_k) \neq 0\}$$

then a generalized solution does not exist. If

$$z_0 \in H_\theta^{4m-2n} \cap \operatorname{span}\{\varphi_k : P_n(\lambda_k) \neq 0\}$$

then there exists a classical solution to (4.1)–(4.3).

Proof. If $m > n$, $(-1)^{m-n} \operatorname{Re}(d_m/c_n) \le 0$, and the spectrum of $\sigma(A_1)$ does not contain common roots of the polynomials P_n and Q_m then M is strongly $(L,0)$-radial by Theorem 5.1 in [17]. In this case

$$\sigma^L(M) = \left\{ \mu_k = \frac{Q_m(\lambda_k)}{P_n(\lambda_k)} : P_n(\lambda_k) \neq 0 \right\}$$

and, hence, we can take

$$a = \sup_{P_n(\lambda_k)\neq 0} \operatorname{Re} \frac{Q_m(\lambda_k)}{P_n(\lambda_k)}$$

in the definition of strong $(L,0)$-radiality.

It is proven in [17] that in our situation we have

$$P = \sum_{P_n(\lambda_k)\neq 0} \langle \cdot, \varphi_k \rangle_{L_2(\Omega)} \varphi_k, \quad Q = \sum_{P_n(\lambda_k)\neq 0} \langle \cdot, \varphi_k \rangle_{L_2(\Omega)} \varphi_k$$

(convergence of the series is understood in the sense of the norm of the space $\mathfrak{U}$ for the operator P and the space $\mathfrak{V}$ for the operator Q). Moreover, $\mathfrak{U}^1$ and $\mathfrak{V}^1$ are the closures of the same set $\operatorname{span}\{\varphi_k : P_n(\lambda_k) \neq 0\}$ in the norm of the spaces $\mathfrak{U}$ or $\mathfrak{V}$, respectively. Finally, we refer to Theorem 3.1 and indicate that the norms $\|Mz_0\|_{L_2(\Omega)}$ and $\|z_0\|_{H^{2m}(\Omega)}$ are equivalent here. □

Example 4.1. Let

$$P_1(\lambda) = 1 + \lambda, \quad Q_2(\lambda) = \lambda + 2\lambda^2, \quad \Omega = (0,\pi), \quad \theta = 0.$$

Then

$$\lambda_k = -k^2, \quad \varphi_k(x) = \sin kx, \quad k \in \mathbb{N}, \quad \sup_{k=2,3,\ldots} \frac{2k^4 - k^2}{1 - k^2} = -9\frac{1}{3} < 0,$$

and, hence, the problem

$$\int_0^\infty z(x,t)\eta(t)\,dt = z_0(x), \quad x \in (0,\pi),$$

$$z(0,t) = \frac{\partial^2 z}{\partial x^2}(0,t) = z(\pi,t) = \frac{\partial^2 z}{\partial x^2}(\pi,t) = 0, \quad t \geq 0,$$

$$\left(1 + \frac{\partial^2}{\partial x^2}\right)\frac{\partial z}{\partial t}(x,t) = \left(\frac{\partial^2}{\partial x^2} + 2\frac{\partial^4}{\partial x^4}\right)z(x,t), \quad (x,t) \in (0,\pi) \times [0,\infty),$$

meets all requirements of Theorem 4.1.

Remark 4.1. Theorem 4.1 remains valid for $m \leq n$. In this case the condition $(-1)^{m-n}\operatorname{Re}(d_m/c_n) \leq 0$ can be omitted, M is continuous on $H_\theta^{2n}(\Omega)$, and the values of k in (4.3) do not exceed $n-1$.

Remark 4.2. The following equations are particular cases of (4.2) [3]: the equation of transient processes in semiconductors

$$(\lambda - \Delta)\frac{\partial z}{\partial t}(x,t) = \alpha z(x,t),$$

the filtration equation in a fractured porous medium [18]

$$(\lambda - \Delta)\frac{\partial z}{\partial t}(x,t) = \alpha\Delta z(x,t),$$

the equation of motion of underground water with free boundary [19]

$$(\lambda - \Delta)\frac{\partial z}{\partial t}(x,t) = \alpha\Delta z(x,t) - \beta\Delta^2 z(x,t).$$

Remark 4.3. In the arguments of this section we can replace the Laplace operator A_1 in $L_2(\Omega)$ with a selfadjoint elliptic operator generally of higher order, i.e. with the operator

$$(A_1 u)(x) = \sum_{|\alpha| \leq 2r} a_\alpha(x) D^\alpha u(x), \quad a_\alpha \in C^\infty(\overline{\Omega}).$$

with the domain $D(A_1) = H^{2r}_{\{B_l\}}(\Omega)$ (see the notations in [20]), where

$$(B_l u)(x) = \sum_{|\alpha| \leq r_l} b_{l\alpha}(x) D^\alpha u(x), \quad b_{l\alpha} \in C^\infty(\partial\Omega), \quad l = 1, 2, \ldots, r,$$

under the conditions of regular ellipticity of the collection $A, B_1, B_2, \ldots, B_r$ [20] and the lower boundedness of the spectrum $\sigma(A_1)$.

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V. E. Fedorov
Laboratory of Quantum Topology, Chelyabinsk State University, Chelyabinsk, Russia
`kar@csu.ru`

N. D. Ivanova
South Ural State University, Chelyabinsk, Russia
`natalia.d.ivanova@gmail.com`

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INVERSE PROBLEMS OF RECOVERING A SOURCE FUNCTION IN HEAT AND MASS TRANSFER SYSTEMS

E. M. Korotkova and S. G. Pyatkov

Abstract. We consider the well-posedness in the Sobolev spaces of the problem of recovering a source function in heat and mass transfer systems. The overdetermination conditions are the values of concentration of admixtures on given surfaces (or separate points). A local existence theorem is proven and some stability estimates for solutions are established.

Keywords: parabolic system, inverse problem, heat and mass transfer, Navier–Stokes system, boundary value problem

Intriduction

We examine the system

$$u_t - \nu\Delta u + (u, \nabla)u + \nabla p = f + \beta_c C + \beta_\theta \Theta, \quad \operatorname{div} u = 0, \tag{1}$$

$$\Theta_t - \lambda_\theta \Delta\Theta + (u, \nabla)\Theta = f_\theta, \tag{2}$$

$$C_t + (u, \nabla)C - \sum_{i,j=1}^{n} a_{ij} C_{x_i x_j} + \sum_{i=1}^{n} a_i C_{x_i} + a_0 C = f_c, \tag{3}$$

where $\nu = \text{const} > 0$, $(x,t) \in Q = G \times (0,T)$ ($G \subset \mathbb{R}^n$, $T < \infty$), u, Θ, p, and C are the velocity vector, the temperature of a fluid, the pressure, the vector of concentration of admixtures (organic or inorganic) in a fluid, and f_c is the volume density of admixture sources, respectively. Here a_{ij}, a_i, and a_0 are matrices of size $h \times h$, with h the number of admixtures, β_C is a matrix of size $n \times h$, β_Θ is a vector of length n, and $\lambda_\Theta > 0$ is a scalar function. System (1)–(3) describes the admixture propagation in a fluid. In particular, the classical Oberbeck–Boussinesq model (see [1–4]) can be written in the form of (1)–(3). The functions f_θ and f are the densities of the heat sources and exterior forces, respectively, and λ_θ is the heat conductivity coefficient. In the Oberbeck–Boussinesq model β_c and β_θ are the mass and heat transfer coefficients multiplied by the gravity acceleration. Here we assume that β_c is an arbitrary matrix-function of size $n \times h$ and β_θ is a vector-function of length n.

The system (1)–(3) is complemented with the initial and boundary conditions

$$u|_{t=0} = u_0, \quad u|_S = g_1(t,x), \quad \Gamma = \partial G,\ S = \Gamma \times (0,T), \tag{4}$$

$$\Theta|_{t=0} = \Theta_0, \quad \Theta|_S = g_2(t,x), \quad C|_{t=0} = C_0, \quad C|_S = g_3(t,x). \tag{5}$$

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We consider the inverse problem on determining a solution to (1)–(3) and the right-hand side f_c in (3) with the use of additional measurements on sections of G or at distinguished points.

Let $x'' = (x_{s+1}, x_{s+2}, \dots, x_n)$ $(s = 0, 1, \dots, n-1)$. If $s \geq 1$ then we denote $x' = (x_1, x_2, \dots, x_s)$. We assume that the right-hand side of (3) is known in some domain $Q' = G_1 \times (0, T)$ and unknown in the domain $Q'' = G_0 \times (0, T)$, where G_1 and G_0 are either nonempty disjoint domains such that $\overline{G_0} \cup \overline{G_1} = \overline{G}$ or $G_1 = \varnothing$ and so $Q'' = Q$. The right-hand side of (3) is of the form

$$f_c = f_0(x,t) + \sum_{i=1}^{m} f_i(x,t) q_i(x',t), \quad (x,t) \in Q, \tag{6}$$

where f_i $(i = 0, 1, \dots, m)$ are given functions vanishing on Q'. The scalar functions $q_i(x', t)$ in this representation are unknown and can be found on using the overdetermination conditions:

$$C|_{S_i} = \psi_i(t, x') \quad (S_i = (0,T) \times \Gamma_i,\ i = 1, 2, \dots, r,\ m = rh), \tag{7}$$

where $\{\Gamma_i\}$ is a collection of smooth s-dimensional surfaces lying in G. For $s = 0$ these surfaces Γ_i are just points in $G_0 = G$.

The inverse problems of this type appear in chemistry, biology, and other fields while describing heat and mass transfer processes, diffusion, and filtration. Numerical methods for solving boundary value problems for (1)–(3) are collected in [4]. We refer to [5], where many inverse and extremal problems for (1)–(3) in the stationary case are exhibited. Similar problems in a simplified setting are considered in [6–10]. Note that in the real models of use are the regional decision support systems, several equations are used for different admixtures in a fluid even in the one-dimensional case. The parameters taken into account are phytoplankton, apiphyton, and various chemical compounds. Many coefficient inverse problems with the overdetermination conditions of the form (6) and $s = n - 1$ for second order parabolic equations are considered in the articles by Yu. Ya. Belov, Yu. E. Anikonov, and other authors (see [11]). In the case of $n = 1$ ($s = 0$, the unknowns q_i depend only on t and the surfaces S_i are points) such linear and nonlinear problems are treated in [12] for second order parabolic equations. Among the recent articles we point out [13–15], where some analogs of problems (1)–(6) are examined for parabolic systems of equations. The well-posedness of inverse problems for parabolic equations with the overdetermination conditions of the form (7) (including numerical methods) are presented in [16–21]. We note the monographs [22–25] which are devoted to inverse problems for parabolic and elliptic equations and systems of equations, where statements of the problems and some results can be found.

1. Notations and Auxiliary Statements

Let E be a Banach space. Denote by $L_p(G; E)$ (with G a domain in $\mathbb{R}^n$) the space of strongly measurable functions on G with values in E which is endowed with the norm $\|\|u(x)\|_E\|_{L_p(G)}$ (see, for instance, [26, Section 1.18.4]). We also use the spaces $C^k(\overline{G}; E)$ of functions having the derivatives up to the order k continuous and bounded on G and admitting a continuous extensions to $\overline{G}$. The definitions of the Sobolev spaces $W_p^k(G; E)$ and $W_p^k(Q; E)$ are conventional (see [26]). If $E = \mathbb{C}$ or $E = \mathbb{C}^n$ then we use the notation $W_p^k(G)$ or $C^k(\overline{G})$. The membership $u \in W_p^k(G)$ (or $u \in C^k(\overline{G})$) for a given vector-function $u = (u_1, u_2, \dots, u_k)$ means that every component u_i belongs to $W_p^k(G)$ (or $C^k(\overline{G})$). The norm on the corresponding space is the sum of the norms of coordinates, unless otherwise stated. A similar convention is used for matrices; i.e., the membership $a \in W_p^k(G)$ $(a = \{a_{ij}\}_{j,i=1}^k)$ means that $a_{ij}(x) \in W_p^k(G)$ for all i and j. Given an interval $J = (0,T)$, put $W_p^{k,r}(Q) = W_p^r(J; L_p(G)) \cap L_p\big(J; W_p^k(G)\big)$ and $W_p^{k,r}(S) = W_p^r(J; L_p(\Gamma)) \cap L_p\big(J; W_p^k(\Gamma)\big)$. Denote by $L_{p,\sigma}(G)$ the closure of solenoidal C_0^∞-vector-functions with respect to the norm of $L_p(G)$ and put $W_{p,\sigma}^k(G) = W_p^k(G) \cap L_{p,\sigma}(G)$ and $W_{p,\sigma}^{k,k/2}(Q) = W_p^{k,k/2}(Q) \cap L_p(0,T; L_{p,\sigma}(G))$ $(k \geq 0)$. The symbol $\overset{\circ}{W}{}_q^k(G)$ stands for the closure of $C_0^\infty(G)$ with respect to the norm of $W_q^k(G)$ and $\dot{W}_q^1(G) = \{p \in L_{q,loc}(G) : \nabla p \in L_q(G)\}$. We identify the functions differing by a constant and endow the last space with the norm $\|p\|_{\dot{W}_q^1(G)} = \|\nabla p\|_{L_q(G)}$. It is a Banach space. The notation $\nabla_{x''} f(x,t)$ designates the vector-function $(\partial_{x_{s+1}} f, \partial_{x_{s+2}} f, \dots, \partial_{x_n} f)$, where ∂_{x_k} denotes the partial derivative $\frac{\partial}{\partial x_k}$.

Describe the class of domains G. We say that the boundary $\Gamma = \partial G$ *belongs to* C^β $(\beta \geq 1)$ if there exist numbers $d, r > 0$ such that, for every $x_0 \in \Gamma$, there exists a neighborhood U about x_0 with the following properties: in the local coordinate system y, obtained by rotation and translation of the original system so that the axis y_n is directed along the inner normal to Γ at x_0, we have

$$\overline{U} \cap G = \{y \in \mathbb{R}^n : y' \in \overline{B_r}, \omega(y') < y_n \leq \omega(y') + d\}, \quad y' = (y_1, \dots, y_{n-1}),$$
$$\overline{U} \cap (\mathbb{R}^n \setminus \overline{G}) = \{y \in \mathbb{R}^n : \omega(y') - d \leq y_n < \omega(y')\},$$
$$\Gamma \cap \overline{U} = \{y \in \mathbb{R}^n : y' \in \overline{B_r},\ y_n = \omega(y')\},$$

where $y_n = \omega(y')$ is the equation of Γ, $\omega \in C^\beta(\overline{B_r})$ $(B_r = \{y' : |y'| < r\})$ and the norms of all functions ω in $C^\beta(\overline{B_r})$ are bounded by a constant independent of x_0. Without loss of generality we may assume that $Mr < d/4$, with M the Lipschitz constant of ω in B_r. Write down the conditions on G_0 and Γ_i.

(A) (a) The case of $s > 0$. There exists a domain $\Omega \subset \mathbb{R}^s$ with boundary of class C^2 such that $G_0 \subset \Omega \times \mathbb{R}^{n-s}$,

$$\Gamma_i = \big\{x \in \mathbb{R}^n : x'' = \varphi^i(x') = \big(\varphi_{s+1}^i(x'), \varphi_{s+2}^i(x'), \dots, \varphi_n^i(x')\big), x' \in \Omega\big\},$$

$\varphi^i(x') \in C^2(\overline{\Omega})$, and there exists a constant $\delta > 0$ such that

$$U_{\delta i} = \{(x', \varphi^i(x') + \eta) : x' \in \Omega,\ \eta \in \mathbb{R}^{n-s},\ |\eta| < \delta\} \subset G_0, \quad \rho(U_{\delta i}, G \setminus G_0) > 0,$$

for $i = 1, 2, \dots, r$ and $U_{\delta i} \cap U_{\delta j} = \varnothing$ for $i \neq j$, $i, j = 1, 2, \dots, r$.

(b) The case of $s = 0$. In this case the sets $\{\Gamma_i\}_{i=1}^r$ are interior points $\{x_i\}_{i=1}^r$ of G. Put $U_{\delta i} = B_\delta(x_i)$ and choose a number $\delta > 0$ such that $\overline{U_{\delta i}} \subset G$ and $U_{\delta i} \cap U_{\delta j} = \varnothing$ for $i \neq j$, $i, j = 1, 2, \dots, r$.

Condition (A) is used in all articles on the problems in question. As is easily seen, it ensures uniqueness of solutions. Condition (A) is fulfilled if $G_0 = G = \Omega \times \mathbb{R}^{n-s}$, with Ω a bounded or unbounded domain of class C^2. In what follows we use the notations $Q^\tau = (0,\tau) \times G$, $Q_0^\tau = (0,\tau) \times \Omega$, $Q_T = (0,T) \times \Omega$, $G_\delta = \cup_i U_{\delta i}$, $Q_\tau^{\delta i} = (0,\tau) \times U_{\delta i}$, and $Q_\tau^\delta = (0,\tau) \times G_\delta$.

Lemma 1. *Let $u \in W_q^{2,1}(Q^\tau)$ $(q \in (1,\infty))$ and let $u(x,0) = 0$. Then there exists a constant $c > 0$ independent of u such that*

$$\|u\|_{L_q(Q^\tau)} \leq c\tau \|u\|_{W_q^{2,1}(Q^\tau)}, \quad \|\nabla u\|_{L_q(Q^\tau)} \leq c\tau^{1/2} \|u\|_{W_q^{2,1}(Q^\tau)}.$$

The claim results from the Newton–Leibnitz formula and the interpolation inequality $\|\nabla u\|_{L_q(G)} \le c\|u\|^{1/2}_{W_q^2(G)}\|u\|^{1/2}_{L_q(G)}$.

The following lemma ensues from Lemma 1 and Lemma 3.3 in Chapter 2 of [27], where $\delta = \sqrt{\tau}$.

Lemma 2. *Let $u \in W_q^{2,1}(Q^\tau)$. Then $u \in L_p(Q^\tau)$ for $2 - (\frac{1}{q} - \frac{1}{p})(n+2) \ge 0$ and $p \ge q$ and $\nabla u \in L_p(Q^\tau)$ for $1 - (\frac{1}{q} - \frac{1}{p})(n+2) \ge 0$ and $p \ge q$. Moreover, $u \in C^{\lambda,\lambda/2}(\overline{Q^\tau})$ for $q > (n+2)/2$ and $\nabla u \in C^{\lambda,\lambda/2}(\overline{Q^\tau})$ for $q > n+2$, where $\lambda \in [0, 2-(n+2)/q)$ in the former case and $\lambda \in [0, 1-(n+2)/q)$ in the latter. The following estimates are valid for the corresponding values of parameters:*

$$\|u\|_{L_p(Q^\tau)} \le c\tau^{\beta_1}\|u\|_{W_q^{2,1}(Q^\tau)}, \quad \|\nabla u\|_{L_p(Q^\tau)} \le c\tau^{\beta_1 - 1/2}\|u\|_{W_q^{2,1}(Q^\tau)},$$

$$\|u\|_{C^{\lambda,\lambda/2}(\overline{Q^\tau})} \le c\tau^{\beta_2}\|u\|_{W_q^{2,1}(Q^\tau)}, \quad \|\nabla u\|_{C^{\lambda,\lambda/2}(\overline{Q^\tau})} \le c\tau^{\beta_2 - 1/2}\|u\|_{W_q^{2,1}(Q^\tau)},$$

where $\beta_1 = 1 - \frac{(n+2)}{2}(\frac{1}{q} - \frac{1}{p})$, $\beta_2 = 1 - \frac{(n+2)}{2q} - \frac{\lambda}{2}$, and c is a constant independent of $\tau \le T$ and $u \in W_q^{2,1}(Q^\tau)$.

Lemma 3. *Let $b \in L_p(Q)$. Then the inequalities are valid for $\tau \in (0,T]$: if $q > (n+2)/2$ and $p \ge q$ then*

$$\|bu\|_{L_q(Q^\tau)} \le c\tau^{1-\frac{n+2}{2p}}\|u\|_{W_q^{2,1}(Q^\tau)};$$

if $q > n+2$ and $p \ge q$ then

$$\|b\nabla u\|_{L_q(Q^\tau)} \le c\tau^{1/2-\frac{(n+2)}{2p}}\|u\|_{W_q^{2,1}(Q^\tau)}.$$

The constant $c > 0$ is independent of $\tau \le T$ and $u \in W_q^{2,1}(Q^\tau)$.

The proof of this lemma is contained in that of Theorem 9.1 of Chapter 4 in [27].

Theorem 1. *For every $f \in L_r(Q)$, $r \in (1,\infty)$, there exist a unique vector-function $u \in W_{r,\sigma}^{2,1}(Q) \cap L_r\big(0,T;\overset{\circ}{W}{}_r^1(G)\big)$ and a function $p \in L_r(0,\tau;\dot{W}_r^1(G))$ such that*

$$u_t - \nu\Delta u + \nabla p = f, \quad \operatorname{div} u = 0, \quad u|_S = 0, \quad u|_{t=0} = 0$$

and

$$\|u\|_{W_r^{2,1}(Q)} + \|\nabla p\|_{L_r(Q)} \le c\|f\|_{L_r(Q)},$$

with c a constant independent of f.

Corollary 1. *For every $f \in L_r(Q^\tau)$, $\tau \in (0,T]$, there exist a unique vector-function $u \in W_{r,\sigma}^{2,1}(Q^\tau) \cap L_r\big(0,\tau;\overset{\circ}{W}{}_r^1(G)\big)$ and a function $p \in L_r\big(0,\tau;\dot{W}_r^1(G)\big)$ such that*

$$u_t - \nu\Delta u + \nabla p = f, \quad \operatorname{div} u = 0, \quad u|_S = 0, \quad u|_{t=0} = 0 \tag{8}$$

and

$$\|u\|_{W_r^{2,1}(Q^\tau)} + \|\nabla p\|_{L_r(Q^\tau)} \le c\|f\|_{L_r(Q^\tau)},$$

with c a constant independent of f and τ.

The theorem follows from Theorem 1.1 in [28]. We can refer also to Theorem 1.2 in [29] and the properties of the Helmholtz projection.

The next result will be a theorem on solvability of parabolic problems. We examine the problem

$$u_t - Lu = f, \quad u|_S = 0, \quad u(x,0) = 0, \tag{9}$$

where

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_ix_j} - \sum_{i=1}^{n} a_i(x,t)u_{x_i} - a_0(x,t)u,$$

a_{ij}, a_i, and a_0 are matrices of size $h \times h$, and there exists a constant $\delta_1 > 0$ such that

$$\sum_{i,j=1}^{n} (a_{ij}(x,t)\xi^i, \xi^j) \geq \delta_1 \sum_{i=1}^{n} \|\xi^i\|^2 \quad \forall \xi^j \in \mathbb{R}^h,\ (x,t) \in Q,\ j = 1,2,\dots,n. \tag{10}$$

We suppose that

$$a_{ij} \in C(\overline{Q}), \quad a_i \in L_q(Q), \quad a_0 \in L_q(Q), \quad i,j = 1\dots n. \tag{11}$$

Theorem 2. *Assume that $\partial G \in C^2$, $q > n+2$, $\gamma \in (0,T]$, and conditions* (10) *and* (11) *hold. Then, for every $f \in L_q(Q^\gamma)$, there exists a unique solution $u \in W_q^{2,1}(Q^\gamma)$ to* (9) *satisfying*

$$\|u\|_{W_q^{2,1}(Q^\gamma)} \leq c\|f\|_{L_q(Q^\gamma)},$$

with c a constant independent of $\gamma \in (0,T)$.

For a fixed parameter $\gamma = T$, the theorem results from Theorem 10.4 of Chapter 7 in [27] (see also [30]). The independence on γ is obvious.

Theorem 3. *Assume that $\partial G \in C^2$ and conditions* (A), (10), *and* (11) *are fulfilled. We require also that*

$$\nabla_{x''} f \in L_q\big(Q_\gamma^\delta\big), \quad \nabla_{x''} a_{ij} \in L_\infty\big(Q_T^\delta\big), \quad \nabla_{x''} a_i, \in L_q\big(Q_T^\delta\big), \quad \nabla_{x''} a_0 \in L_q\big(Q_T^\delta\big)$$

for some $q > n+2$ and all $i,j = 1,2,\dots,n$.

Then a solution $u \in W_q^{2,1}(Q^\gamma)$ to problem (9) *possesses the properties: $\nabla_{x''} u \in W_q^{2,1}\big(Q_\gamma^{\delta_1}\big)$ for every $\delta_1 < \delta$ and*

$$\|\nabla_{x''} u\|_{W_q^{2,1}(Q_\gamma^{\delta_1})} \leq c(\|f\|_{L_q(Q^\gamma)} + \|\nabla_{x''} f\|_{L_q(Q_\gamma^\delta)}).$$

The constant c here depends on $\delta_1 < \delta$ and is independent of $\gamma \leq T$.

2. The Main Results

First, we impose some conditions on the data, assuming that condition (A) is fulfilled. In the sequel, $q > n+2$.

Agreement and smoothness conditions can be written as follows: There exist vector-functions Φ_1, Φ_3, and Φ_2 such that

$$\Phi_i(t,x) \in W_q^{2,1}(Q) : \Phi_1|_{t=0} = u_0,\ \Phi_2|_{t=0} = \Theta_0,\ \Phi_3|_{t=0} = C_0,\ \Phi_i|_S = g_i, \tag{12}$$

$$\operatorname{div} \Phi_1 = 0, \quad \Phi_3|_{S_j} = \psi_j, \quad f_0, f_\theta, f \in L_q(Q),\ f_j \in L_\infty(Q), \tag{13}$$

$$\nabla_{x''}\Phi_3 \in W_q^{2,1}\big(Q_T^\delta\big), \quad \nabla_{x''} f_0 \in L_q\big(Q_T^\delta\big), \quad \nabla_{x''} f_j \in L_\infty\big(Q_T^\delta\big), \tag{14}$$

where $j = 1,2,\dots,m$, $i = 1,2,3$, and δ is the constant from (A).

Define the matrix B as follows: the rows with the numbers from $(k-1)h+1$ to kh $(k = 1,2,\dots,r)$ are occupied by the columns

$$[f_1(x', \varphi^k(x'), t), f_2(x', \varphi^k(x'), t), \dots, f_m(x', \varphi^k(x'), t)].$$

We assume that there exists a constant $\delta_0 > 0$ such that

$$|\det B| \geq \delta_0 > 0, \quad \text{a.a. in } Q_T, \tag{15}$$

and the conditions hold:

(B) $\lambda_\theta(x,t) \geq \delta_1 > 0\ \forall (x,t) \in Q$, $\lambda_\theta, a_{ij} \in C(\overline{Q})$, and $\nabla_{x''} a_{ij} \in L_\infty(Q_T^\delta)$ for all $i,j = 1,2,\dots,n$; β_c, a_i, a_0, $\beta_\theta \in L_q(Q)$, $\nabla_{x''} a_i$, $\nabla_{x''} a_0 \in L_q(Q_T^\delta)$, $i = 1,2,\dots,n$.

Theorem 4. *Assume that $\Gamma \in C^2$, $q > n+2$ and conditions* (A), (B), (10), *and* (12)–(15) *hold. Fix $R_0 > 0$. Then there exists a number $\tau_0 = \tau_0(R_0) \in (0, T]$ such that for all data $(\Phi_1, \Phi_2, \Phi_3, f, f_\theta, f_0)$ satisfying the condition*

$$\sum_{j=1}^{3}(\|\Phi_j\|_{W_q^{2,1}(Q)} + \|\nabla_{x''}\Phi_3\|_{W_q^{2,1}(Q_T^\delta)} + \|f\|_{L_q(Q)}$$
$$+\|f_\theta\|_{L_q(Q)} + \|f_0\|_{L_q(Q)} + \|\nabla_{x''} f_0\|_{L_q(Q_T^\delta)}) \le R_0 \tag{16}$$

there exists a unique solution $(u, p, \Theta, C, q_1, \dots, q_m)$ to (1)–(7) *of the class*

$$u \in W_q^{2,1}(Q^{\tau_0}),\ p \in L_q(0, \tau_0; \dot W_q^1(G)),\ q_j \in L_q(Q_0^{\tau_0})\ j = 1, 2, \dots, m,$$
$$\Theta, C \in W_q^{2,1}(Q^{\tau_0}), \quad \nabla_{x''} C \in W_q^{2,1}\big(Q_{\tau_0}^{\delta_2}\big) \quad \forall \delta_2 < \delta.$$

For a given $\delta_1 < \delta$, there exists a constant $c = c(R_0, \delta_1)$ such that every pair of solutions u^i, Θ^i, C^i, q^i, $q^i = (q_{i1}, q_{i2}, \dots, q_{im})$, $i = 1, 2$, from this class with the data $(\Phi_1^i, \Phi_2^i, \Phi_3^i, f^i, f_\theta^i, f_0^i)$, $i = 1, 2$, satisfies the estimate

$$\|u^1 - u^2\|_{W_q^{2,1}(Q^{\tau_0})} + \|\Theta^1 - \Theta^2\|_{W_q^{2,1}(Q^{\tau_0})} + \|C^1 - C^2\|_{W_q^{2,1}(Q^{\tau_0})}$$
$$+\|\nabla_{x''}(C^1 - C^2)\|_{W_q^{2,1}(Q_{\tau_0}^{\delta_1})} + \sum_{j=1}^{m} \|q_{1j} - q_{2j}\|_{L_q(Q_0^{\tau_0})}$$
$$\le c\bigg(\sum_{j=1}^{3}\big(\big\|\Phi_j^1 - \Phi_j^2\big\|_{W_q^{2,1}(Q^{\tau_0})} + \big\|\nabla_{x''}\Phi_3^1 - \nabla_{x''}\Phi_3^2\big\|_{W_q^{2,1}(Q_{\tau_0}^\delta)} + \|f^1 - f^2\|_{L_q(Q^{\tau_0})}$$
$$+\big\|f_\theta^1 - f_\theta^2\big\|_{L_q(Q^{\tau_0})} + \big\|f_0^1 - f_0^2\big\|_{L_q(Q^{\tau_0})} + \big\|\nabla_{x''} f_0^1 - \nabla_{x''} f_0^2\big\|_{L_q(Q_{\tau_0}^\delta)}\bigg).$$

3. Proof of the Main Results

Proof of Theorem 4. Make the change of variables $u = v + \Phi_1$, $\Theta = \Theta_1 + \Phi_2$, and $C = C_1 + \Phi_3$. We obtain

$$L_{01}(v, p) = v_t - \nu\Delta v + \nabla p = g + \beta_c C_1 + \beta_\theta \Theta_1$$
$$-(\Phi_1, \nabla)v - (v, \nabla)v - (v, \nabla)\Phi_1, \quad \operatorname{div} v = 0, \tag{17}$$
$$L_{02}\Theta_1 = \Theta_{1t} - \lambda_\theta \Delta\Theta_1 = g_\theta - (v, \nabla)\Theta_1 - (\Phi_1, \nabla)\Theta_1 - (v, \nabla)\Phi_2, \tag{18}$$
$$L_{03}C_1 = C_{1t} - \sum_{i,j=1}^{n} a_{ij} C_{1x_ix_j} + \sum_{j=1}^{n} a_j C_{1x_j} + a_0 C_1 = g_c$$
$$-(v, \nabla)C_1 - (\Phi_1, \nabla)C_1 - (v, \nabla)\Phi_3 + \sum_{j=1}^{m} f_j q_j,$$

where the new function g_θ and the vector-functions g and g_c are of the form

$$g = f - \Phi_{1t} + \nu\Delta\Phi_1 - (\Phi_1, \nabla)\Phi_1 + \beta_c\Phi_3 + \beta_\theta\Phi_2,\ g_\theta = f_\theta - L_{02}\Phi_2 - (\Phi_1, \nabla)\Phi_2,$$
$$g_c = f_0 - L_{03}\Phi_3 - (\Phi_1, \nabla)\Phi_3.$$

Denote by Φ_{1j}, $j = 1, 2, \dots, n$, the coordinates of Φ_1. The new functions v, θ_1, and the vector-function C_1 satisfy the homogeneous boundary conditions (4), (5), and (7). Next, we determine q_{0i} as a solution to the system

$$\sum_{i=1}^{m} f_i(x', \varphi^j(x'), t) q_{0i} + g_c(x', \varphi^j(x'), t) = 0, \quad j = 1, 2, \dots, r. \tag{20}$$

Equality (20) can be written as

$$Bq_0 = -\vec{g},$$

where the matrix B is defined in Section 2. The coordinates of the vector $\vec{g}$ from $h(k-1)+1$ to hk coincide with those of the vector-function $\vec{g_c}(x', \varphi^k(x'), t)$. In view of (15) B is invertible.

Note that the vector-function g_c belongs to $L_q(Q)$ and $\nabla_{x''} g_c \in L_q(Q_T^\delta)$. In this case the trace $g_c|_{\Gamma_i} \in L_q(Q_T)$ is correctly defined. Find q_{0i} from (20) and put $q_i = q_{0i} + q_{1i}$. We obtain the equation

$$L_{03}C_1 = C_{1t} - \sum_{i,j=1}^{n} a_{ij} C_{1x_ix_j} + \sum_{j=1}^{n} a_j C_{1x_j} + a_0 C_1 = g_{0c} - (v, \nabla)C_1$$
$$-(\Phi_1, \nabla)C_1 - (v, \nabla)\Phi_3 + \sum_{j=1}^{m} f_j q_{1j}, \quad g_{0c} = g_c + \sum_{j=1}^{m} f_j q_{0j}. \tag{21}$$

We arrive at an equivalent problem. Let $\gamma \in (0, T]$. By Theorems 1 and 2, we can rewrite (17), (18), and (21) as follows:

$$(v, p) = (L_{01})^{-1} g + (L_{01})^{-1}(\beta_c C_1 + \beta_\theta \Theta_1 - (\Phi_1, \nabla)v - (v, \nabla)v - (v, \nabla)\Phi_1), \tag{22}$$

$$\Theta_1 = (L_{02})^{-1} g_\theta - (L_{02})^{-1}((v, \nabla)\Theta_1 + (\Phi_1, \nabla)\Theta_1 + (v, \nabla)\Phi_2), \tag{23}$$

$$C_1 = (L_{03})^{-1} g_{0c} + (L_{03})^{-1}\Big(-(v, \nabla)C_1 - (\Phi_1, \nabla)C_1 - (v, \nabla)\Phi_3 + \sum_{j=1}^{m} f_j q_{1j}\Big). \tag{24}$$

Here the operator $(L_{01})^{-1}$ takes $g \in L_q(Q^\gamma)$ into the pair (v, p) presenting a solution to the equation $L_{01}(v, p) = g$ and such that $\operatorname{div} v = 0$, $v \in W_q^{2,1}(Q^\gamma)$, $p \in L_q(0, \gamma; \dot{W}_q^1(G))$, and the vector-function v satisfies the homogeneous initial and boundary conditions. The operators $(L_{0i})^{-1}$, $i = 2, 3$, are defined similarly with the use of Theorem 2. Let $(L_{01})^{-1} g = (v_0, p_0)$. Define the space H^γ comprising the vectors (v, p, Θ, C), where $v \in W_q^{2,1}(Q^\gamma)$ is a solenoidal vector-function of length n satisfying the homogeneous conditions (4), $C, \Theta \in W_q^{2,1}(Q^\gamma)$ are a vector of length h and a scalar function, respectively, satisfying the homogeneous conditions (5), and p is a scalar function in $L_q\big(0, \gamma; \dot{W}_q^1(G)\big)$. Endow this space with the norm

$$\|(v, p, \Theta, C)\|_{H^\gamma} = \|v\|_{W_q^{2,1}(Q^\gamma)} + \|p\|_{L_q(0,\gamma;\dot{W}_q^1(G))} + \|\Theta\|_{W_q^{2,1}(Q^\gamma)} + \|C\|_{W_q^{2,1}(Q^\gamma)}.$$

Let $R = 3\|v_0, p_0, (L_{02})^{-1} g_\theta, (L_{03})^{-1} g_{0c}\|_{H^T}$. We look for a solution to system (22)–(24) which can be rewritten as

$$\omega = A(\omega, q^1), \quad \omega = (v, p, \Theta, C), \quad q^1 = (q_{11}, q_{12}, \ldots, q_{1m}), \tag{25}$$

where A is defined by the right-hand side of (22)–(24). Assume that $\omega \in B_{R,\gamma} = \{\omega \in H^\gamma : \|\omega\|_{H^\gamma} \le R\}$. In view of (13) and Theorem 2, we infer

$$\Big\|(L_{03})^{-1}\Big(\sum_{j=1}^{m} f_j q_{1j}\Big)\Big\|_{W_q^{2,1}(Q^\gamma)} \le c\Big\|\sum_{j=1}^{m} f_j q_{1j}\Big\|_{L_q(Q^\gamma)} \le c_1 \|q^1\|_{L_q(Q_0^\gamma)}. \tag{26}$$

Assume that $c_1\|q^1\|_{L_q(Q_0^\gamma)} \le R/3$; i.e., the vector-function q^1 belongs to the closed ball $B^\gamma_{R/(3c_1)}$ of radius $R/(3c_1)$ centered at the origin of $L_q(Q_0^\gamma)$. Estimate $\|A(\omega,q^1)\|_{H^\gamma}$. Theorems 1 and 2 yield

$$\|A(\omega,q^1)\|_{H^\gamma} \le c\Big(\|\beta_c C_1 + \beta_\theta\Theta_1 - (\Phi_1,\nabla)v - (v,\nabla)v - (v,\nabla)\Phi_1\|_{L_q(Q^\gamma)}$$
$$+\|(v,\nabla)\Theta_1 + (\Phi_1,\nabla)\Theta_1 + (v,\nabla)\Phi_2)\|_{L_q(Q^\gamma)}$$
$$+\Big\|(v,\nabla)C_1 + (\Phi_1,\nabla)C_1 + (v,\nabla)\Phi_3 - \sum_{j=1}^m f_j q_{1j}\Big\|_{L_q(Q^\gamma)}\Big) + R/3.$$

Estimate each of the summands as follows:

$$\|\beta_c C_1\|_{L_q(Q^\gamma)} \le \|\beta_c\|_{L_q(Q^T)}\|C_1\|_{L_\infty(Q^\gamma)} \le c_1\gamma^{\beta_1}\|C_1\|_{W_q^{2,1}(Q^\gamma)}.$$

By analogy,

$$\|\beta_\theta\Theta_1\|_{L_q(Q^\gamma)} \le c_2\gamma^{\beta_2}\|\Theta_1\|_{W_q^{2,1}(Q^\gamma)}.$$

The next summand is estimated as

$$\|(\Phi_1,\nabla)v\|_{L_q(Q^\gamma)} \le \|\Phi_1\|_{L_q(Q^T)}\|\nabla v\|_{L_\infty(Q^\gamma)} \le c_3\gamma^{\beta_3}\|v\|_{W_q^{2,1}(Q^\gamma)}.$$

Similarly,

$$\|(\Phi_1,\nabla)\Theta_1\|_{L_q(Q^\gamma)} \le c_4\gamma^{\beta_4}\|\Theta_1\|_{W_q^{2,1}(Q^\gamma)},\ \|(\Phi_1,\nabla)C_1\|_{L_q(Q^\gamma)} \le c_5\gamma^{\beta_5}\|C_1\|_{W_q^{2,1}(Q^\gamma)}.$$

We have

$$\|(v,\nabla)\Phi_1\|_{L_q(Q^\gamma)} \le \|\nabla\Phi_1\|_{L_q(Q^T)}\|v\|_{C(Q^\gamma)} \le c_6\gamma^{\beta_6}\|v\|_{W_q^{2,1}(Q^\gamma)}.$$

Moreover,

$$\|(v,\nabla)\Phi_2\|_{L_q(Q^\gamma)} \le c_7\gamma^{\beta_7}\|v\|_{W_q^{2,1}(Q^\gamma)}, \quad \|(v,\nabla)\Phi_3\|_{L_q(Q^\gamma)} \le c_8\gamma^{\beta_8}\|v\|_{W_q^{2,1}(Q^\gamma)}.$$

At last, we arrive at the inequality

$$\|(v,\nabla)v\|_{L_q(Q^\gamma)} \le \|\nabla v\|_{C(Q^\gamma)}\|v\|_{C(Q^\gamma)} \le c_9\gamma^{\beta_9}\|v\|_{W_q^{2,1}(Q^\gamma)}.$$

We obtain

$$\|(v,\nabla)\Theta_1\|_{L_q(Q^\gamma)} \le c_{10}(R)\gamma^{\beta_{10}}\|\Theta_1\|_{W_q^{2,1}(Q^\gamma)}$$

and

$$\|(v,\nabla)C_1\|_{L_q(Q^\gamma)} \le c_{11}(R)\gamma^{\beta_{11}}\|C_1\|_{W_q^{2,1}(Q^\gamma)}.$$

Summing all summands and taking (26) into account, we infer

$$\|A(\omega,q^1)\|_{H^\gamma} \le \frac{R}{3} + c_1(R)\gamma^{\beta_1}\|v\|_{W_q^{2,1}(Q^\gamma)}$$
$$+c_2(R)\gamma^{\beta_2}\|\Theta_1\|_{W_q^{2,1}(Q^\gamma)} + c_3(R)\gamma^{\beta_3}\|C_1\|_{W_q^{2,1}(Q^\gamma)} + c_1\|q^1\|_{L_q(Q_0^\gamma)}.$$

Since $\gamma \in (0,T]$, choosing $\beta = \min(\beta_1,\beta_2,\beta_3)$, we have

$$\|A(\omega,q^1)\|_{H^\gamma} \le R/3 + c_0(R)\gamma^\beta\|\omega\|_{H^\gamma} + c_1\|q^1\|_{L_q(Q_0^\gamma)}, \tag{27}$$

where the constant c_1 is independent of R and β. Choose $\gamma_0 \le T$ such that $c_0(R)\gamma^\beta \le R/3\ \forall\gamma \le \gamma_0$. In this case (27) can be rewritten as

$$\|A(\omega,q^1)\|_{H^\gamma} \le R \quad \forall q^1 \in B^\gamma_{R/(3c_1)}.$$

The latter means that, for every $q^1 \in B^{\gamma}_{R/(3c_1)}$, the operator $A(\omega, q^1)$ maps the ball $B_{R,\gamma}$ into itself. Similar arguments validate an estimate for $\|A(\omega^1, q^1)-A(\omega^2, q^1)\|_{H^\gamma}$, where $\omega^i = (v^i, p^i, \Theta^i, C^i)$, $i = 1, 2$. We obtain

$$\|A(\omega^1, q^1) - A(\omega^2, q^1)\|_{H^\gamma} \leq c(\|\beta_c(C^1 - C^2) + \beta_\theta(\Theta^1 - \Theta^2)$$
$$-(\Phi_1, \nabla)(v^1 - v^2) - (v^1, \nabla)v^1 + (v^2, \nabla)v^2 - (v^1 - v^2, \nabla)\Phi_1\|_{L_q(Q^\gamma)}$$
$$+\|((v^1, \nabla)\Theta^1 - (v^2, \nabla)\Theta^2) + (\Phi_1, \nabla)(\Theta^1 - \Theta^2) + (v^1 - v^2, \nabla)\Phi_2\|_{L_q(Q^\gamma)}$$
$$+\|(v^1, \nabla)C^1 - (v^2, \nabla)C^2 + (\Phi_1, \nabla)(C^1 - C^2) + (v^1 - v^2, \nabla)\Phi_3\|_{L_q(Q^\gamma)}).$$

Estimate each of the summands as follows:

$$\|\beta_c(C^1 - C^2)\|_{L_q(Q^\gamma)} \leq c_1\gamma^{\beta_1}\|(C^1 - C^2)\|_{W_q^{2,1}(Q^\gamma)}.$$
$$\|\beta_\theta(\Theta^1 - \Theta^2)\|_{L_q(Q^\gamma)} \leq c_2\gamma^{\beta_2}\|(\Theta^1 - \Theta^2)\|_{W_q^{2,1}(Q^\gamma)}.$$

We have

$$\|(\Phi_1, \nabla)(v^1 - v^2)\|_{L_q(Q^\gamma)} \leq \|\Phi_1\|_{L_q(Q^T)}\|\nabla(v^1 - v^2)\|_{C(\overline{Q^\gamma})} \leq c_3\gamma^{\beta_3}\|v^1 - v^2\|_{W_q^{2,1}(Q^\gamma)}.$$

By analogy, we infer

$$\|(\Phi_1, \nabla)(\Theta^1 - \Theta^2)\|_{L_q(Q^\gamma)} \leq c_4\gamma^{\beta_4}\|\Theta^1 - \Theta^2\|_{W_q^{2,1}(Q^\gamma)}$$

and

$$\|(\Phi_1, \nabla)(C^1 - C^2)\|_{L_q(Q^\gamma)} \leq c_5\gamma^{\beta_5}\|C^1 - C^2\|_{W_q^{2,1}(Q^\gamma)}.$$

Estimate the summand

$$\|(v^1 - v^2, \nabla)\Phi_1\|_{L_q(Q^\gamma)} \leq \|\nabla\Phi_1\|_{L_q(Q^T)}\|v^1 - v^2\|_{C(\overline{Q^\gamma})} \leq c_6\gamma^{\beta_6}\|v^1 - v^2\|_{W_q^{2,1}(Q^\gamma)}.$$

Similarly,

$$\|(v^1 - v^2, \nabla)\Phi_2\|_{L_q(Q^\gamma)} \leq c_7\gamma^{\beta_7}\|v^1 - v^2\|_{W_q^{2,1}(Q^\gamma)}$$

and

$$\|(v^1 - v^2, \nabla)\Phi_3\|_{L_q(Q^\gamma)} \leq c_8\gamma^{\beta_8}\|v^1 - v^2\|_{W_q^{2,1}(Q^\gamma)}.$$

Accounting for the relations

$$(v^1, \nabla)v^1 - (v^2, \nabla)v^2 = (v^1 - v^2, \nabla)v^1 + (v^2, \nabla)(v^1 - v^2),$$
$$\|(v^1 - v^2, \nabla)v^1\|_{L_q(Q^\gamma)} \leq \|\nabla v^1\|_{C(Q^\gamma)}\|v^1 - v^2\|_{C(Q^\gamma)} \leq c(R)\gamma^\beta\|v^1 - v^2\|_{W_q^{2,1}(Q^\gamma)},$$
$$\|(v^2, \nabla)(v^1 - v^2)\|_{L_q(Q^\gamma)} \leq \|v^2\|_{C(Q^\gamma)}\|\nabla(v^1 - v^2)\|_{C(Q^\gamma)} \leq c(R)\gamma^\beta\|v^1 - v^2\|_{W_q^{2,1}(Q^\gamma)},$$

we derive the inequality

$$\|(v^1, \nabla)v^1 - (v^2, \nabla)v^2\|_{L_q(Q^\gamma)} \leq c_9(R)\gamma^{\beta_9}\|v^1 - v^2\|_{W_q^{2,1}(Q^\gamma)}.$$

Similarly, we deduce that

$$\left\|(v^1, \nabla)\Theta_1^1 - (v^2, \nabla)\Theta_1^2\right\|_{L_q(Q^\gamma)} \leq c_{10}(R)\gamma^{\beta_{10}}\left\|\Theta_1^1 - \Theta_1^2\right\|_{W_q^{2,1}(Q^\gamma)}$$

and

$$\|(v^1, \nabla)C^1 - (v^2, \nabla)C^2\|_{L_q(Q^\gamma)} \leq c_{11}(R)\gamma^{\beta_{11}}\|C^1 - C^2\|_{W_q^{2,1}(Q^\gamma)}.$$

Choosing an appropriate number β, we justify the inequality

$$\|A(\omega^1, q^1) - A(\omega^2, q^1)\|_{H^\gamma} \leq c_2(R)\gamma^\beta\|\omega^1 - \omega^2\|_{H^\gamma};$$

i.e., A is contractive for $c_2(R)\gamma^\beta = r_0 < 1$. In particular,

$$\|A(\omega^1, q^1) - A(0, q^1)\|_{H^\gamma} \le c_2(R)\gamma^\beta \|\omega^1\|_{H^\gamma}.$$

Fix a constant $r_0 < 1$ and find a constant $\gamma_1 \le \gamma_0$ such that $c_2(R)\gamma^\beta \le r_0$ for $\gamma \le \gamma_1$. The fixed point theorem implies that, for every $q^1 \in B^\gamma_{R/(3c_1)}$ with $\gamma \le \gamma_1$, system (25) and system (22)–(24) respectively have a unique solution ω in $B_{R,\gamma}$. A solution meets the estimate

$$\|\omega\|_{H^\gamma} \le \|A(0, q^1)\|_{H^\gamma} + \|A(\omega, q^1) - A(0, q^1)\|_{H^\gamma} \le R/3 + c_1\|q^1\|_{L_q(Q_0^\gamma)} + r_0\|\omega\|_{H^\gamma}.$$

This inequality means that

$$\|\omega\|_{H^\gamma} \le \frac{R}{3(1-r_0)} + \frac{c_1}{1-r_0}\|q^1\|_{L_q(Q_0^\gamma)}. \tag{28}$$

By Theorem 3, a solution C_1 to (24) possesses the property $\nabla_{x''}C_1 \in W_p^{2,1}\big(Q_\gamma^{\delta_1}\big)$ and the corresponding inequality of the theorem holds. In particular, (28) yields an estimate for C_1 of the form

$$\|\nabla_{x''}C_1\|_{W_q^{2,1}(Q_\gamma^{\delta_1})} \le c\|q^1\|_{L_q(Q_0^\gamma)} + c_0(R), \tag{29}$$

where the constant $\delta_1 < \delta$ is fixed and $\gamma \le \gamma_1$.

Consider two vectors $q^1, q^2 \in B^\gamma_{R/(3c_1)}$ and find two solutions ω^1 and ω^2 ($\omega^i = (v^i, p^i, \Theta^i, C^i)$, $i = 1, 2$) to (22)–(24). Their difference $\omega^1 - \omega^2$ satisfies the equality

$$\omega^1 - \omega^2 = A(\omega^1, q^1) - A(\omega^2, q^1) + A(\omega^2, q^1) - A(\omega^2, q^2)$$

which implies that

$$\|\omega^1 - \omega^2\|_{H^\gamma} \le r_0\|\omega^1 - \omega^2\|_{H^\gamma} + c_1\|q^1 - q^2\|_{L_q(Q_0^\gamma)}$$

and

$$\|\omega^1 - \omega^2\|_{H^\gamma} \le \frac{c_1}{1-r_0}\|q^1 - q^2\|_{L_q(Q_0^\gamma)}. \tag{30}$$

Let $\gamma \le \gamma_1$. Equation (24) can be written in the form of (21). Note that the right-hand side of (21) satisfies the conditions of Theorem 3. By Theorem 3, solutions C^i, $i = 1, 2$, to (24) are such that $\nabla_{x''}C^i \in W_p^{2,1}\big(Q_\gamma^{\delta_1}\big)$ and the corresponding inequality of the theorem holds. Subtracting two equations (21) relating to q^1 and q^2 and using Theorem 3 and (30), we obtain an estimate for the difference $C^1 - C^2$ of the form

$$\|\nabla_{x''}(C^1 - C^2)\|_{W_q^{2,1}(Q_\gamma^{\delta_1})} \le c\|q^1 - q^2\|_{L_q(Q_0^\gamma)}, \tag{31}$$

where $\delta_1 < \delta$ is fixed.

Fix a number $l = 1, 2, \ldots, r$. Make the change of variables $y'' = x'' - \varphi^l(x')$, $y' = x'$ in (21) in the domain $U_{\delta_1 l}$, with $\delta_1 < \delta$. Describe the connections between the derivatives in old and new variables. We have

$$\partial_{x_j} = \partial_{y_j} - \sum_{r=s+1}^{n} \varphi^l_{ry_j}(y')\partial_{y_r} \quad (j \le s), \quad \partial_{x_j} = \partial_{y_j} \quad (j > s),$$

$$\partial_{y_j} = \partial_{x_j} + \sum_{r=s+1}^{n} \varphi^l_{rx_j}(x')\partial_{x_r} \quad (j \le s), \quad \partial_{y_j} = \partial_{x_j} \quad (j > s).$$

Next, we infer

$$v^i(y,t), C^i(y,t) \in W_q^{2,1}(Q_{\gamma,\delta_1}), \quad \nabla_{y''} C^i \in W_q^{2,1}(Q_{\gamma,\delta_1}), \quad i = 1,2,$$

where $Q_{\gamma,\delta_1} = \Omega \times G^{\delta_1} \times (0,\gamma)$ and $G^{\delta_1} = (-\delta_1, \delta_1)^{n-s}$. The system (20) can be rewritten as

$$C_{1t} - L^l C_1 = g_{0c} - \sum_{j=1}^{n} \alpha_j^l v_j - \sum_{j=1}^{n} \beta_j^l C_{1y_j} + \sum_{j=1}^{m} f_j q_{1j}, \quad i = 1,2, \tag{32}$$

where α_j^l is a linear combination of coordinates of the vectors $\nabla_y \Phi_3$, while β_j^l is spanned by the vectors Φ_1 and v,

$$L^l C_1 = \sum_{i,j=1}^{n} a_{ij}^l(y,t) C_{1y_iy_j} - \sum_{i=1}^{n} a_i^l(y,t) C_{1y_i} - a_0^l(y,t) C_1,$$

and the operator L^l satisfies the conditions of Theorems 2 and 3. Denote by L_1^l the part of L^l not containing the derivatives with respect to $y_{s+1},\dots,y_n$ and by L_2^l, the difference $L^l - L_1^l$.

Since all summands in (32) and their derivatives with respect to $y_{s+1},\dots,y_n$ belong to $L_q(Q_{\gamma,\delta_1})$ and $q > n+2$, their traces exist at $y'' = 0$. Take $y'' = 0$. In result, we derive the relation

$$\begin{aligned} S_{0l}(q^1) &= \Big(-L_2^l C_1 + \sum_{j=1}^{n} \alpha_j^l v_j + \sum_{j=s+1}^{n} \beta_j^l C_{1y_j}\Big)\Big|_{y''=0} \\ &= \sum_{j=1}^{m} f_j(y', \varphi^l(y'), t) q_{1j}(y',t), \quad i = 1,2, \end{aligned} \tag{33}$$

which can be rewritten as

$$q^1 = B^{-1} S_0(q^1) = S(q^1), \tag{34}$$

where the coordinates of S_0 from $(l-1)h+1$ to lh coincide with the coordinates of S_{0l}. Here the right-hand side can be treated as an operator S over the vector-functions $q^1 \in B_{R/3c_1}^{\gamma}$ with $\gamma \le \gamma_1$. The functions $C_1, v_1, \dots, v_n$ of the vector $S(q^1)$ are expressed through q^1 by means of (22)–(24).

Demonstrate that (34) has a unique solution locally in time. Let $q^1, q^2 \in B_{R/(3c_1)}^{\gamma}$. The properties of the matrix B yield

$$\|S(q^1) - S(q^2)\|_{L_q(Q_0^\gamma)} \le c \sum_{l=1}^{m} \|S_{0l}(C^1, v^1) - S_{0l}(C^2, v^2)\|_{L_q(Q_0^\gamma)}, \tag{35}$$

where C^i, v^i ($v^i = (v_1^i, v_2^i, \dots, v_n^i)$, $i = 1,2$) are solutions corresponding to q^1 and q^2 in (22)–(24). Write down the difference $S_{0l}(C^1, v^1) - S_{0l}(C^2, v^2)$ as follows:

$$S_{0l}(C^1,v^1) - S_{0l}(C^2,v^2) = -L_2^l(C^1 - C^2) + \sum_{j=1}^{n} \alpha_j^l (v_j^1 - v_j^2) + \sum_{j=s+1}^{n} (\beta_j^{1l} C^1_{y_j} - \beta_j^{2l} C^2_{y_j}).$$

We have

$$L_2^l(C^1 - C^2) = \sum_{j=s+1}^{n} \sum_{i=1}^{n} a_{ij}^l (C^1_{y_iy_j} - C^2_{y_iy_j}) + \sum_{i \ge s+1} a_i^l (C^1_{y_i} - C^2_{y_i}). \tag{36}$$

Estimate each of the summands separately. Put $B_{\delta_1} = \{y'' : |y''| < \delta_1\}$. We have

$$\begin{aligned}
&\Big\| \sum_{j=s+1}^{n} \sum_{i=1}^{n} a^l_{ij} \big(C^1_{y_i y_j} - C^2_{y_i y_j}\big)\big|_{y''=0} \Big\|_{L_q(Q_0^\gamma)} \\
&\le c_1 \sum_{j=s+1}^{n} \sum_{i=1}^{n} \big\| \big(C^1_{y_i y_j} - C^2_{y_i y_j}\big)\big|_{y''=0} \big\|_{L_q(Q_0^\gamma)} \\
&\le c_2 \sum_{j=s+1}^{n} \sum_{i=1}^{n} \big\| \big(C^1_{y_i y_j} - C^2_{y_i y_j}\big) \big\|_{W_q^\alpha(B_{\delta_1}; L_q(Q_0^\gamma))},
\end{aligned}$$

where $\alpha \in (\frac{n-s}{q}, 1)$ (see the embedding theorems in [31, Chapter 6, Section 6.1]). Applying the interpolation inequality, Lemma 1, and estimate (31), we infer

$$\begin{aligned}
&c_2 \sum_{j=s+1}^{n} \sum_{i=1}^{n} \big\| \big(C^1_{y_i y_j} - C^2_{y_i y_j}\big) \big\|_{W_q^\alpha(B_{\delta_1}; L_q(Q_0^\gamma))} \\
&\le c_3 \sum_{j=s+1}^{n} \sum_{i=1}^{n} \big\| C^1_{y_i y_j} - C^2_{y_i y_j} \big\|^{\alpha}_{W_q^1(B_{\delta_1}; L_q(Q_0^\gamma))} \big\| C^1_{y_i y_j} - C^2_{y_i y_j} \big\|^{1-\alpha}_{L_q(B_{\delta_1}; L_q(Q_0^\gamma))} \\
&\le c_4 \gamma^{\frac{1-\alpha}{2}} \|\nabla_{y''}(C^1 - C^2)\|_{W_q^{2,1}(B_{\delta_1} \times Q_0^\gamma)} \le c_5 \gamma^{\beta_5} \|q^1 - q^2\|_{L_q(Q_0^\gamma)}.
\end{aligned}$$

Estimating the second summand in (36), we derive that

$$\Big\| \sum_{i=s+1}^{n} a^l_i \big(C^1_{y_i} - C^2_{y_i}\big)\big|_{y''=0} \Big\|_{L_q(Q_0^\gamma)} \le c_2 \sum_{i=s+1}^{n} \big\| a^l_i \big(C^1_{y_i} - C^2_{y_i}\big) \big\|_{W_q^1(B_{\delta_1}; L_q(Q_0^\gamma))}.$$

Taking it into account that

$$\frac{\partial}{\partial y_k} a^l_i \big(C^1_{y_i} - C^2_{y_i}\big)_{y_i} = a^l_{i_{y_k}} (C^1 - C^2)_{y_i} + a^l_i (C^1 - C^2)_{y_i y_k},$$

we infer that

$$\begin{aligned}
&c_2 \sum_{i=s+1}^{n} \big\| a^l_i \big(C^1_{y_i} - C^2_{y_i}\big) \big\|_{W_q^1(B_{\delta_1}; L_q(Q_0^\gamma))} \\
&\le c_3 \left(\sum_{i=s+1}^{n} \big\| a^l_i \nabla_y \big(C^1_{y_i} - C^2_{y_i}\big) \big\|_{L_q(B_{\delta_1}; Q_0^\gamma)} + \sum_{i=s+1}^{n} \big\| a^l_{i_{y_i}} \big(C^1_{y_i} - C^2_{y_i}\big) \big\|_{L_q(B_{\delta_1}; Q_0^\gamma)} \right).
\end{aligned}$$

The inequalities of Lemma 3 imply that

$$\begin{aligned}
&c_3 \left(\sum_{i=s+1}^{n} \big\| a^l_i \nabla_y \big(C^1_{y_i} - C^2_{y_i}\big) \big\|_{L_q(B_{\delta_1}; Q_0^\gamma)} + \sum_{i=s+1}^{n} \big\| a^l_{i_{y_i}} \big(C^1_{y_i} - C^2_{y_i}\big) \big\|_{L_q(B_{\delta_1}; Q_0^\gamma)} \right) \\
&\le c_4 \gamma^{\beta_5} \|\nabla_{y''}(C^1 - C^2)\|_{W_q^{2,1}(B_{\delta_1}; Q_0^\gamma)} \le c_5 \gamma^{\beta_6} \|q^1 - q^2\|_{L_q(Q_0^\gamma)}.
\end{aligned}$$

Finally, we have

$$\big\| L^l_2 (C^1 - C^2)|_{y''=0} \big\|_{L_q(Q_0^\gamma)} \le c_1 \gamma^{\beta_7} \|q^1 - q^2\|_{L_q(Q_0^\gamma)}$$

for some $\beta_7 > 0$ and $c_1 > 0$. Next,

$$\Big\|\sum_{j=1}^{n}\alpha_j^l\big(v_j^1-v_j^2\big)\big|_{y''=0}\Big\|_{L_q(Q_0^\gamma)} \leq c\Big\|\sum_{j=1}^{n}\alpha_j^l\big(v_j^1-v_j^2\big)\Big\|_{W_q^1(B_{\delta_1};L_q(Q_0^\gamma))}$$
$$\leq \bigg(\Big\|\sum_{j=1}^{n}\alpha_{jy_k}^l\big(v_j^1-v_j^2\big)\Big\|_{L_q(B_{\delta_1}\times Q_0^\gamma)} + \Big\|\sum_{j=1}^{n}\alpha_j^l\big(v_j^1-v_j^2\big)_{y_k}\Big\|_{L_q(B_{\delta_1}\times Q_0^\gamma)}\bigg)$$
$$\leq c_4\gamma^{\tau_1}\|v^1-v^2\|_{W_q^{2,1}(B_{\delta_1}\times Q_0^\gamma)} + c_5\gamma^{\tau_2}\|v^1-v^2\|_{W_q^{2,1}(B_{\delta_1}\times Q_0^\gamma)}.$$

Passing to the variables x, we validate the estimate

$$\Big\|\sum_{j=1}^{n}\alpha_j^l\big(v_j^1-v_j^2\big)\big|_{y''=0}\Big\|_{L_q(Q_0^\gamma)} \leq c\gamma^{\beta_2}\|v^1-v^2\|_{W_q^{2,1}(Q^\gamma)} \leq c_2\gamma^{\beta_2}\|q^1-q^2\|_{L_q(Q_0^\gamma)},$$

with $\beta_2 = \min(\tau_1,\tau_2)$. The last summand is estimated as

$$\Big\|\sum_{j=s+1}^{n}\big(\beta_j^{1l}C_{y_j}^1-\beta_j^{2l}C_{y_j}^2\big)\big|_{y''=0}\Big\|_{L_q(Q_0^\gamma)}.$$

We have

$$\beta_j^{1l}C_{y_j}^1-\beta_j^{2l}C_{y_j}^2 = \big(\beta_j^{1l}-\beta_j^{2l}\big)C_{y_j}^1+\beta_j^{2l}\big(C_{y_j}^1-C_{y_j}^2\big).$$

Hence,

$$\big\|\big(\beta_j^{1l}-\beta_j^{2l}\big)C_{y_j}^1+\beta_j^{2l}\big(C_{y_j}^1-C_{y_j}^2\big)\big\|_{L_q(Q_0^\gamma)}$$
$$\leq \big\|\big(\beta_j^{1l}-\beta_j^{2l}\big)C_{y_j}^1\big\|_{L_q(Q_0^\gamma)}+\|\beta_j^{2l}\big(C_{y_j}^1-C_{y_j}^2\big)\big\|_{L_q(Q_0^\gamma)}$$
$$\leq c_1\big\|\beta_j^{1l}-\beta_j^{2l}\big\|_{C(\overline{Q_0^\gamma})}\|C_{y_j}^1\|_{L_q(Q_0^\gamma)}+c_2\big\|\beta_j^{2l}\big\|_{C(\overline{Q_0^\gamma})}\big\|C_{y_j}^1-C_{y_j}^2\big\|_{L_q(Q_0^\gamma)}$$
$$\leq c_1\gamma^{\tau_1}\big\|\beta_j^{1l}-\beta_j^{2l}\big\|_{W_q^{2,1}(Q^\gamma)}+c_2(R)\gamma^{\tau_2}\|C^1-C^2\|_{W_q^{2,1}(Q^\gamma)}$$
$$\leq c_1\gamma^{\tau_1}\|q^1-q^2\|_{L_q(Q_0^\gamma)}+c_2(R)\gamma^{\tau_2}\|q^1-q^2\|_{L_q(Q_0^\gamma)} \leq c_3(R)\gamma^{\beta_3}\|q^1-q^2\|_{L_q(Q_0^\gamma)},$$

where $\beta_3 = \min(\tau_1,\tau_2)$. Summing all summands, we infer

$$\|S(q^1)-S(q^2)\|_{L_q(Q_0^\gamma)} \leq c_1\gamma^{\beta_1}\|q^1-q^2\|_{L_q(Q_0^\gamma)}$$
$$+c_2\gamma^{\beta_2}\|q^1-q^2\|_{L_q(Q_0^\gamma)}+c_3(R)\gamma^{\beta_3}\|q^1-q^2\|_{L_q(Q_0^\gamma)}.$$

For $\beta_0 = \min(\beta_1,\beta_2,\beta_3)$, we derive finally that

$$\|S(q^1)-S(q^2)\|_{L_q(Q_0^\gamma)} \leq c(R)\gamma^{\beta_0}\|q^1-q^2\|_{L_q(Q_0^\gamma)}. \tag{37}$$

Obtain an additionally estimate for $\|S(0)\|_{L_q(Q_0^\gamma)}$. Repeating the arguments in the proof of (37) and taking (28) and (29) into account, we arrive at the estimate

$$\|S(0)\|_{L_q(Q_0^\gamma)} \leq \gamma^{\beta_0}c_1(R), \quad \gamma\leq\gamma_1. \tag{38}$$

Rewrite (34) as

$$q^1 = S(0)+(S(q^1)-S(0)),$$

By (37)

$$\|S(q^1)-S(0)\|_{L_q(Q_0^\gamma)} \leq c(R)\gamma^{\beta_0}\|q^1\|_{L_q(Q_0^\gamma)}, \quad \gamma\leq\gamma_1.$$

In view of (38), there exists $\gamma_2\leq\gamma_1$ such that

$$\gamma^{\beta_0}c_1(R)\leq R/(6c_1), \quad \gamma^{\beta_0}c(R)\leq 1/2 \quad \forall\gamma\leq\gamma_2.$$

In this case, for all $\gamma \in (0,\gamma_2]$ and $q^1 \in B^{\gamma}_{R/3c_1}$, we have

$$\|S(q^1)\|_{L_q(Q_0^\gamma)} = \|S(0) + (S(q^1) - S(0))\|_{L_q(Q_0^\gamma)}$$

$$\le \|S(0)\|_{L_q(Q_0^\gamma)} + \|(S(q^1) - S(0))\|_{L_q(Q_0^\gamma)} \le \frac{R}{6c_1} + \frac{1}{2}\|q^1\|_{L_q(Q_0^\gamma)} \le \frac{R}{6c_1} + \frac{R}{6c_1} \le \frac{R}{3c_1},$$

i.e. the operator S takes the ball $B^{\gamma_2}_{R/(3c_1)}$ into itself and as contractive in this ball. The fixed point theorem implies that (34) has the unique solution q^1 in this ball. Take $\tau_0 = \gamma_2$. Define the vector-function ω as the solution to (22)–(24). It satisfies the homogeneous conditions (4) and (5). Demonstrate that (7) holds. Fix $l = 1, 2, \dots, r$. After the change of variables $x \to y$ in (21), we infer

$$C_{1t} - L^l C_1 = g_{0c} - \sum_{j=1}^{n} \alpha_j^l v_j - \sum_{j=1}^{n} \beta_j^l C_{1y_j} + \sum_{j=1}^{m} f_j q_{1j}. \tag{39}$$

Taking the trace at $y'' = 0$, we can rewrite (39) as

$$\Big(C_{1t} - L^l C_1 + \sum_{j=1}^{n} \alpha_j^l v_j + \sum_{j=1}^{n} \beta_j^l C_{1y_j}\Big)\Big|_{y''=0} = \sum_{j=1}^{m} f_j(y', \varphi^l(y'), t) q_{1j}(y', t). \tag{40}$$

In view of (33), we have

$$\widetilde{C}_t - L_1^l \widetilde{C}_t + \sum_{j=1}^{s} \beta_j^l C_{1y_j} = 0, \quad \widetilde{C} = C_1(y', 0, t). \tag{41}$$

Moreover, $\widetilde{C}(y', 0, 0) = 0$ and $\widetilde{C}|_{\partial\Omega\times(0,\gamma)} = 0$. Uniqueness of solutions implies that $C(y', 0, t) = 0$.

Proceed with the proof of stability estimates. Fix R_0. Obviously, R, defined in the proof, satisfies the condition $R \le cR_0$ (c is some constant). Choosing cR_0 as R and repeating the arguments, we see that the parameter γ_1 chosen in the proof is the same for all data from our class and thus estimates (28) and (29) hold with the constants independent of the data from our class. Repeating the proof of existence, we find that the interval of solvability is the same for all data. It depends only on R. A solution q^1 lies in the ball of radius $R/3c_1$ and the norms od solutions, as it follows from (28) and (29) are bounded by the same constant depending on R, i.e.,

$$\|\omega\|_{H^\gamma} + \|\nabla_{x''} C_1\|_{W_q^{2,1}(Q_\gamma^{\delta_1})} \le c_0(R), \tag{42}$$

where $\delta_1 < \delta$ is fixed. Take two solutions corresponding to the two different collections $(C^i, \Theta^i, v^i, q^i)$ $(v^i = (v_1^i, v_2^i, \dots, v_n^i),\ i = 1, 2)$ of the data. Each of them satisfies (22)–(24), where on the right-hand side the functions g^i, g_θ^i, and g_{0c}^i are used rather than g, g_θ, and g_{0c}. Subtracting the systems, we can estimate the norm of the difference of solutions and obtain the estimate

$$\begin{aligned}\|\omega^1 - \omega^2\|_{H^\gamma} + \|\nabla_{x''}(C^1 - C^2)\|_{W_q^{2,1}(Q_\gamma^{\delta_1})} &\le c_1(\|q^1 - q^2\|_{L_q(Q_0^\gamma)} \\ + \|g^1 - g^2\|_{L_q(Q^\gamma)} + \left\|g_\theta^1 - g_\theta^2\right\|_{L_q(Q^\gamma)} &+ \left\|g_{0c}^1 - g_{0c}^2\right\|_{L_q(Q^\gamma)}).\end{aligned} \tag{43}$$

rather than (30) and (31). Next, we repeat the arguments employed in the proof of (37) and (38). Consider (33) written for these two solutions. Subtracting them from one another and repeating the above arguments, we arrive at the estimates

$$\|q^1 - q^2\|_{L_q(Q_0^\gamma)} \le c(R)\gamma^\beta(\|\omega^1 - \omega^2\|_{H^\gamma} + \|\nabla_{x''}(C^1 - C^2)\|_{W_q^{2,1}(Q_\gamma^{\delta_1})}),$$

with $\beta > 0$ and $c(R)$ positive constants. Replacing the norms on the right-hand side of the last inequality with the use of (43) and choosing $\gamma_3 \le \gamma_2$ sufficiently small, we arrive at the inequality

$$\|q^1 - q^2\|_{L_q(Q_0^\gamma)} \le c_2\big(\|g^1 - g^2\|_{L_q(Q^\gamma)} + \big\|g_\theta^1 - g_\theta^2\big\|_{L_q(Q^\gamma)} + \big\|g_{0c}^1 - g_{0c}^2\big\|_{L_q(Q^\gamma)}\big), \quad \gamma \le \gamma_3.$$

Employing this inequality on the right-hand side of (43), we obtain

$$\|\omega^1 - \omega^2\|_{H^\gamma} + \|\nabla_{x''}(C^1 - C^2)\|_{W_q^{2,1}(Q_\gamma^{\delta_1})}$$
$$\le c_2\big(\|g^1 - g^2\|_{L_q(Q^\gamma)} + \big\|g_\theta^1 - g_\theta^2\big\|_{L_q(Q^\gamma)} + \big\|g_{0c}^1 - g_{0c}^2\big\|_{L_q(Q^\gamma)}\big). \tag{44}$$

The last two estimates ensure the stability estimate for the formulation of the theorem. The theorem is proven.

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E. M. Korotkova
Yugra State University, Khanty-Mansiisk, Russia
Ugra Research Institute of Information Technologies, Khanty-Mansiisk, Russia
`kem@uriit.ru`

S. G. Pyatkov
Yugra State University, Khanty-Mansiisk, Russia
Sobolev Institute of Mathematics, Novosibirsk, Russia
`s-pyatkov@ugrasu.ru; pyatkov@math.nsc.ru`

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AN OPTIMAL TILT ANGLE OF A FLAT CRACK IN THE EQUILIBRIUM PROBLEM FOR THE KIRCHHOFF–LOVE PLATE

N. P. Lazarev

Abstract. We study the equilibrium problem for a plate in the Kirchhoff–Love model with the nonpenetration condition in the form of an inequality for a flat oblique crack. Solvability of the corresponding control problem is proven. The derivative of the quality functional serves as the cost functional and the tilt angles of the plane crack as the control functions.

Keywords: oblique crack, optimal control, Kirchhoff–Love plate, variational inequality

Introduction

We examine the well-known variational statement of the equilibrium problem for an elastic plate with an oblique crack [1]. The rigid clamping condition and the nonpenetration condition of the crack edges are given on the respective exterior and interior boundaries of a domain with a cut corresponding to the middle part of the plate. We prove solvability of the optimal control problem with a parameter characterizing the tilt angle of a flat oblique crack. The derivative of the energy functional with respect to the perturbation parameter of a flat oblique crack found in [2] serves as the quality functional.

The derivative of the energy functional with respect to the length of a crack is often used in the statements of fracture criterions [3]. The problem of differentiation of the energy functional in linear problems is sufficiently well studied (see, for instance, [4, 5]). The articles [6, 7] are devoted to nonlinear problems with the nonpenetration condition in the form of inequalities and the analysis of the behavior of the energy functional and a solution under the perturbation of the length of a crack or the form of a volume. At present, the mathematical models for problems of the crack theory with Signorini-type conditions are well studied (see, for instance, the monographs [7, 8] and the survey [9]). In particular, for the models of elastic two- and three-dimensional bodies and the Timoshenko and Kirchhoff–Love plates, the smoothness properties of solutions are described, the fictitious domain method is justified, invariant integrals were found, the different control problems are studied, and for some problems, the dependence of a solution and physical characteristics of the problem on variation of the elasticity coefficients or the geometry of a domain is analyzed. Moreover, there are some results connected with the mathematical models of inhomogeneous bodies with rigid inclusions (see, for instance, [7, 10, 11]).

1. Statement of the Equilibrium Problem for a Plate

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. Denote by Γ_δ the set $\{(x_1, x_2) \mid 0 < x_1 < l + \delta,\ x_2 = 0\}$, $\delta \in [-\delta_0, \delta_0]$, $l > \delta_0 > 0$, describing the intersection of the crack with the middle plane in the initial undistorted state of the plate.

Assume that $\overline{\Gamma}_{\delta_0} \subset \Omega$. The parameter δ stands for the crack perturbation. Given a fixed $\delta \in [-\delta_0, \delta_0]$, the middle surface of the plate occupies the domain $\Omega_\delta = \Omega \setminus \overline{\Gamma}_\delta$. The domain $\Omega_0 = \Omega \setminus \overline{\Gamma}_0$ corresponds to the unperturbed crack. The middle surface of the plate lies in the plane $z = 0$ and the coordinate system (x_1, x_2, z) is assumed Cartesian. Suppose that the thickness of the plate is equal to 2. The crack is a surface in $\mathbb{R}^3$ described by the relations $x_2 + z \tan\alpha = 0$, $-1 \le z \le 1$, $0 < x_1 < l + \delta$, $\alpha = \text{const}$, $0 \le \alpha \le \alpha_0 < \frac{\pi}{2}$. The number α specifies the tilt angle of the crack. Let $\chi = (W, w)$ be the displacement vector of the points on the middle surface of the plate, with $W = W(x) = (u(x), v(x))$ the horizontal displacement along the middle plane and $w(x)$ the vertical displacement.

The strain tensor of the middle surface of the plate is denoted by [12]:

$$\varepsilon_{ij}(W) = \frac{1}{2}\bigl(w,_j^i + w,_i^j\bigr), \quad w^1 = u, \quad w^2 = v, \quad i, j = 1, 2,$$

where the index after the comma stands for the derivative with respect to the corresponding coordinate. The stress tensors are written as [12]:

$$\sigma_{11}(W) = \varepsilon_{11}(W) + k\varepsilon_{22}(W), \quad \sigma_{22}(W) = \varepsilon_{22}(W) + k\varepsilon_{11}(W),$$

$$\sigma_{12}(W) = \sigma_{21}(W) = (1-k)\varepsilon_{12}(W), \quad k = \text{const}, \ 0 < k < \frac{1}{2}.$$

We assume that the boundary conditions

$$w = \frac{\partial w}{\partial n} = W = 0 \quad \text{on } \partial\Omega \tag{1}$$

hold on the outer boundary, with n the outer normal to $\partial\Omega$. These conditions describe the rigid clamping of a plate.

Let the subspace $H^{1,0}(\Omega_\delta)$ of the Sobolev space $H^1(\Omega_\delta)$ comprise the functions vanishing on $\partial\Omega$. Similarly, the subspace $H^{2,0}(\Omega_\delta)$ of $H^2(\Omega_\delta)$ consists of the functions that vanish on $\partial\Omega$ together with their first derivatives. Put

$$H(\Omega_\delta) = H^{1,0}(\Omega_\delta) \times H^{1,0}(\Omega_\delta) \times H^{2,0}(\Omega_\delta).$$

The nonpenetration condition for oblique cracks can be written as follows [1]:

$$[v] + [w]\tan\alpha \ge |[w,_2]| \quad \text{on } \Gamma_\delta, \tag{2}$$

where $[V] = V^+ - V^-$ stands for the jump of V on Γ_δ, while $V^+ = V|_{\Gamma_\delta^+}$ and $V^- = V|_{\Gamma_\delta^-}$ designate the traces on the positive and negative banks of the cut Γ_δ (in accord with the direction of the axis x_2). For $\alpha = 0$ in (2), we obtain the well-known nonpenetration condition for plates with vertical cracks (see [6–8]). Given parameters $\delta \in [-\delta_0, \delta_0]$ and $\alpha \in [-\alpha_0, \alpha_0]$, examine the sets of admissible displacements

$$K(\alpha, \delta, \Omega_\delta) = \{\chi = (W, w) \in H(\Omega_\delta) \mid \chi \text{ satisfies (2)}\}.$$

For a fixed $\delta \in [-\delta_0, \delta_0]$, we consider the energy functional of a plate

$$\Pi(\Omega_\delta, \chi) = \frac{1}{2}B_\delta(w, w) + \frac{1}{2}\int\limits_{\Omega_\delta} \sigma_{ij}(W)\varepsilon_{ij}(W)\, d\Omega_\delta - \int\limits_{\Omega_\delta} F\chi\, d\Omega_\delta, \tag{3}$$

where $F = (f_1, f_2, f_3) \in C^1(\overline{\Omega})$ is a given vector of external forces and the bilinear form $B_\delta(\cdot,\cdot)$ is defined as

$$B_\delta(w, \overline{w}) = \int_{\Omega_\delta} b(w, \overline{w})\, d\Omega_\delta,$$

with $b(w, \overline{w}) = w_{,11}\,\overline{w}_{,11} + w_{,22}\,\overline{w}_{,22} + kw_{,11}\,\overline{w}_{,22} + kw_{,22}\,\overline{w}_{,11} + 2(1-k)w_{,12}\,\overline{w}_{,12}$. The Korn inequality [13]

$$c_1 \|W\|^2_{H^{1,0}(\Omega_\delta)^2} \le \int_{\Omega_\delta} \sigma_{ij}(W)\varepsilon_{ij}(W)\, d\Omega_\delta$$

and the inequality obtained by the repeated application of the Poincaré inequality [7]

$$c_2 \|w\|^2_{H^{2,0}(\Omega_\delta)} \le B_\delta(w, w)$$

are valid with constants $c_1 > 0$ and $c_2 > 0$ independent of w and W. Basing on the previous two inequalities we establish the equivalence in $H(\Omega_\delta)$ of the conventional norm and the norm defined as

$$\left\{ \int_{\Omega_\delta} \sigma_{ij}(W)\varepsilon_{ij}(W)\, d\Omega_\delta + B_\delta(w, w) \right\}^{\frac{1}{2}}. \tag{4}$$

The equilibrium problem for a plate whose solution satisfies (1) and (2) can be stated as the following minimization problem for the energy functional on the set of all admissible displacements:

$$\min_{\bar{\chi} \in K(\alpha, \delta, \Omega_\delta)} \Pi(\Omega_\delta, \bar{\chi}). \tag{5}$$

As is known, for fixed $\delta \in [-\delta_0, \delta_0]$ and $\alpha \in [-\alpha_0, \alpha_0]$, there exists a unique solution χ_δ^α to (5) [1]. Moreover, (5) is equivalent to the variational inequality (see [1]):

$$\begin{aligned} B_\delta\big(w_\delta^\alpha, \overline{w} - w_\delta^\alpha\big) + \int_{\Omega_\delta} \sigma_{ij}\big(W_\delta^\alpha\big)\varepsilon_{ij}\big(\overline{W} - W_\delta^\alpha\big)\, d\Omega_\delta \\ \ge \int_{\Omega_\delta} F\big(\bar{\chi} - \chi_\delta^\alpha\big)\, d\Omega_\delta, \quad \bar{\chi} \in K(\alpha, \delta, \Omega_\delta). \end{aligned} \tag{6}$$

Note that if a solution to the variational inequality (6) is sufficiently smooth then the problem is equivalent to the differential statement (see [1]):

$$-\sigma_{ij,j} = f_i \quad \text{in } \Omega_\delta, \quad i = 1, 2,$$

$$\Delta^2 w = f_3 \quad \text{in } \Omega_\delta,$$

$$[\sigma_{22}(W)] = [t(w)] = [m(w)] = 0 \quad \text{on } \Gamma_\delta,$$

$$\sigma_{22}(W)\tan\alpha + t(w) = 0, \quad \sigma_{12}(W) = 0, \quad -\sigma_{22}(W) \ge |m(w)| \quad \text{on } \Gamma_\delta,$$

$$[v] + [w]\tan\alpha \ge |[w_{,2}]| \quad \text{on } \Gamma_\delta,$$

$$(-\sigma_{22}(W) - m(w))([v] + [w]\tan\alpha + [w_{,2}]) = 0 \quad \text{on } \Gamma_\delta,$$

$$(-\sigma_{22}(W) + m(w))([v] + [w]\tan\alpha - [w_{,2}]) = 0 \quad \text{on } \Gamma_\delta,$$

where $t(w)$ and $m(w)$ are defined on Γ_δ as follows:

$$t(w) = \frac{\partial}{\partial x_2}\left(\Delta w + (1-\kappa)\frac{\partial^2 w}{\partial x_1^2}\right), \quad m(w) = \kappa\Delta w + (1-\kappa)\frac{\partial^2 w}{\partial x_2^2}.$$

The derivative of the energy functional with respect to the perturbation parameter δ of a crack for the problem (5) is written as (see [2]):

$$G\big(\alpha, \chi_0^\alpha\big) = \frac{d\Pi\big(\Omega_\delta, \chi_\delta^\alpha\big)}{d\delta}\bigg|_{\delta=0} = \lim_{\delta\to 0}\frac{\Pi(\Omega_\delta, \chi_\delta^\alpha) - \Pi(\Omega_0, \chi_0^\alpha)}{\delta}$$

$$= -\frac{1}{2}\int\limits_{\Omega_0}\theta_{,1}\left(\big(u_{0,1}^\alpha\big)^2 - \big(v_{0,2}^\alpha\big)^2 + \frac{1}{2}(1-k)\Big(\big(v_{0,1}^\alpha\big)^2 - \big(u_{0,2}^\alpha\big)^2\Big)\right)d\Omega_0$$

$$-\frac{1}{2}\int\limits_{\Omega_0}\theta_{,2}\left(2v_{0,1}^\alpha v_{0,2}^\alpha + (1+k)v_{0,1}^\alpha u_{0,1}^\alpha + (1-k)u_{0,1}^\alpha u_{0,2}^\alpha\right)d\Omega_0$$

$$-\int\limits_{\Omega_0}\left(w_{0,11}^\alpha P_\alpha^1 + w_{0,22}^\alpha P_\alpha^2 + k\big(w_{0,11}^\alpha P_\alpha^2 + w_{0,22}^\alpha P_\alpha^1\big) + 2(1-k)w_{0,12}^\alpha P_\alpha^3\right)d\Omega_0$$

$$+\frac{1}{2}\int\limits_{\Omega_0}\theta_{,1}\, b\big(w_0^\alpha, w_0^\alpha\big)\,d\Omega_0 - \int\limits_{\Omega_0}(F\theta)_{,1}\,\chi_0^\alpha\,d\Omega_0, \tag{7}$$

where

$$P^1 = 2\theta_{,1}\, w_{0,11}^\alpha + \theta_{,11}\, w_{0,1}^\alpha, \quad P^2 = 2\theta_{,2}\, w_{0,12}^\alpha + \theta_{,22}\, w_{0,1}^\alpha,$$
$$P^3 = \theta_{,1}\, w_{0,12}^\alpha + \theta_{,2}\, w_{0,11}^\alpha + \theta_{,12}\, w_{0,1}^\alpha.$$

The auxiliary function $\theta \in C_0^\infty(\Omega)$ (7) is chosen so that $\theta = 1$ in a neighborhood about $x_l = (l, 0)$, $\theta = 0$ in a neighborhood about $x_0 = (0,0)$, and $\theta_{,2} = 0$ on Γ_{δ_0}. Observe that this function can be used to define some one-to-one correspondence between the domains Ω_0 and Ω_δ of the form

$$y_1 = x_1 - \delta\theta(x_1, x_2), \quad y_2 = x_2,$$

where $y = (y_1, y_2) \in \Omega_0$ and $(x_1, x_2) \in \Omega_\delta$. Moreover, if $\chi(x) \in K(\alpha, \delta, \Omega_\delta)$ is an arbitrary function then the function $\hat\chi(y)$, defined as $\hat\chi(y) = \chi(x)$, with $x = x(y, \delta)$, belongs to $K(\alpha, 0, \Omega_0)$. The reverse inclusion also holds. The membership of $\hat\chi(y)$ in $K(\alpha, 0, \Omega_0)$ implies that $\chi(x) \in K(\alpha, \delta, \Omega_\delta)$ [2].

3. An Optimal Control Problem

For convenience, a solution to (6) corresponding to the parameter $\delta = 0$ is used below without the subscript 0, i.e. $\chi_0^\alpha = \chi^\alpha$. In accord with the results of Section 2, $G(\alpha, \chi^\alpha)$ is given by (7) for all $\alpha \in [-\alpha_0, \alpha_0]$.

Let us state now the optimal control problem: Find $\alpha^* \in [-\alpha_0, \alpha_0]$ satisfying

$$G(\alpha^*, \chi^{\alpha^*}) = \sup_{\alpha\in[-\alpha_0,\alpha_0]} G(\alpha, \chi^\alpha). \tag{8}$$

Theorem. *The optimal control problem* (8) *is solvable.*

Proof. Let $\{\alpha_n\}$ be a maximizing sequence corresponding to (8). Since the interval $[-\alpha_0, \alpha_0]$ is bounded, we can assume, extracting a subsequence if need be, that $\{\alpha_n\}$ tends to some number $\alpha^* \in [-\alpha_0, \alpha_0]$. By the lemma proven below, we can extract a subsequence (preserving the notation) such that $\alpha_n \to \alpha^*$ and $\chi^{\alpha_n} \to \chi^{\alpha^*}$ as $n \to \infty$ strongly in $H(\Omega_0)$. By strong convergence, this sequence satisfies the relation

$$G(\alpha_n, \chi^{\alpha_n}) \to G(\alpha^*, \chi^*) \quad \text{as } n \to \infty.$$

Hence, α^* is a solution to (8). The theorem is proven.

Lemma. *Let $\alpha_n \to \alpha^*$. Then there exists a subsequence of $\{\alpha_n\}$ (we preserve the old notation) such that*

$$\chi^{\alpha_n} \to \chi^{\alpha^*} \quad \text{as } n \to \infty$$

strongly in $H(\Omega_0)$.

Proof. Indeed, we have the relations

$$\int\limits_{\Omega_0} \sigma_{ij}(W^\alpha)\varepsilon_{ij}(W^\alpha)\, d\Omega_0 + B_0(w^\alpha, w^\alpha) = \int\limits_{\Omega_0} F\chi^\alpha\, d\Omega_0, \quad \alpha \in [-\alpha_0, \alpha_0]$$

which easily imply the uniform estimate

$$\|\chi^\alpha\| \le C,$$

where $C > 0$ is independent of $\alpha \in [-\alpha_0, \alpha_0]$. In view of the above estimate and reflexivity of $H(\Omega_0)$, there exist a subsequence of $\{\chi^{\alpha_n}\}$ (we preserve the notation for a new sequence) convergent weakly in $H(\Omega_0)$ to a function $\tilde{\chi}$ as $\alpha_n \to \alpha^*$.

Prove now that $\tilde{\chi} \in K(\alpha^*, 0, \Omega_0)$. In view of the weak convergence of χ^{α_n} to $\tilde{\chi}$ in $H(\Omega_0)$, choosing a subsequence if need be we infer that $\chi^{\alpha_n} \to \tilde{\chi}$ strongly in $L_2(\Gamma_0)^3$ and $w,_2^{\alpha_n} \to \tilde{w},_2^{\alpha^*}$ strongly in $L_2(\Gamma_0)$ as $\alpha_n \to \alpha^*$ (see [14, Theorem 5.19]). Extracting a subsequence once again if need be we can assume that $\chi^{\alpha_n} \to \tilde{\chi}$ and $w,_2^{\alpha_n} \to \tilde{w},_2^{\alpha^*}$ a.a. on Γ_0 as $\alpha_n \to \alpha^*$. Therefore, passing to the limit as $\alpha_n \to \alpha^*$ in the inequalities

$$[v^{\alpha_n}] + [w^{\alpha_n}] \tan\alpha_n \ge \big|[w,_2^{\alpha_n}]\big| \quad \text{on } \Gamma_0,$$

we derive that

$$[\tilde{v}] + [\tilde{w}] \tan\alpha^* \ge |[\tilde{w},_2]| \quad \text{on } \Gamma_0.$$

The latter means that $\tilde{\chi} \in K(\alpha^*, 0, \Omega_0)$.

Let us verify that for every test function $\hat{\eta} = (\widehat{W}, \widehat{w}) \in K(\alpha^*, 0, \Omega_0)$ there exists a sequence $\hat{\eta}^\alpha$ such that $\hat{\eta}^\alpha \in K(\alpha, 0, \Omega_0)$ and $\hat{\eta}^\alpha \to \hat{\eta}$ strongly in $H(\Omega_0)$. It suffices to examine the functions of the form

$$\hat{\eta}^\alpha = (\widehat{W}^\alpha, \widehat{w}^\alpha) = \hat{\eta} + (0, \widehat{w}(\tan\alpha^* - \tan\alpha), 0).$$

It is easy to check that the function constructed satisfies the required properties. Indeed, the inclusion $\eta^\alpha \in H(\Omega_0)$ is obvious. Let us verify the nonpenetration condition. By construction, $[\hat{v}^\alpha] = [\hat{v}] + [\widehat{w}](\tan\alpha^* - \tan\alpha)$, $[\widehat{w},_2^\alpha] = [\widehat{w},_2]$, $[\widehat{w}^\alpha] = [\widehat{w}]$ on Γ_0. Hence, we infer

$$\begin{aligned} [\hat{v}^\alpha] + [\widehat{w}^\alpha]\tan\alpha &= [\hat{v}] + [\widehat{w}](\tan\alpha^* - \tan\alpha) + [\widehat{w}]\tan\alpha \\ &= [\hat{v}] + [\widehat{w}]\tan\alpha^* \ge |[\widehat{w},_2]| = |[\widehat{w},_2^\alpha]| \quad \text{on } \Gamma_0. \end{aligned} \tag{9}$$

The strong convergence $\hat{\eta}^\alpha \to \hat{\eta}$ in $H(\Omega_0)$ is obvious. Thereby, $\{\hat{\eta}^\alpha\}$ has the required properties.

Now we can demonstrate that $\tilde{\chi} = \chi^{\alpha^*}$. To this end, we insert test functions of the form $\hat{\eta}^{\alpha_n}$ in the variational inequalities (6) with $\delta = 0$ corresponding to α_n, $n = 1, 2, \dots$, and pass to the limit as $n \to \infty$. In result, we see that

$$\begin{gathered} B_0(\tilde{w}, \hat{w} - \tilde{w}) + \int\limits_{\Omega_0} \sigma_{ij}(\tilde{W})\varepsilon_{ij}(\widehat{W} - \tilde{W})\, d\Omega_0 \ge \int\limits_{\Omega_0} F(\hat{\eta} - \tilde{\chi}) d\Omega_0, \\ \hat{\eta} = (\widehat{W}, \widehat{w}) \in K(\alpha^*, 0, \Omega_0). \end{gathered} \tag{10}$$

Taking the unique solvability of (10) into account, we have $\tilde{\chi} = \chi^{\alpha^*}$. The weak convergence yields

$$\lim_{n\to\infty} \int\limits_{\Omega_0} \sigma_{ij}(W^{\alpha_n})\varepsilon_{ij}(W^{\alpha_n})\, d\Omega_0 + B_0(w^{\alpha_n}, w^{\alpha_n})$$
$$= \lim_{n\to\infty} \int\limits_{\Omega_0} F\chi^{\alpha_n}\, d\Omega_0 = \int\limits_{\Omega_0} F\chi^{\alpha^*} d\Omega_0. \tag{11}$$

Since the conventional norm and the norm on $H(\Omega_\delta)$ defined by (4) are equivalent, the last equality implies the strong convergence $\chi^{\alpha_n} \to \chi^{\alpha^*}$ in $H(\Omega_0)$ as $n \to \infty$. The lemma is proven.

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N. P. Lazarev
Institute of Mathematics of North-Eastern Federal University, Yakutsk, Russia
nyurgun@ngs.ru

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SINGULAR SOLUTIONS OF THE $(3+1)$–D PROTTER PROBLEM FOR THE WAVE EQUATION

N. Popivanov, T. Popov, and R. Scherer

Abstract. We study some boundary value problems for the nonhomogeneous wave equation with three space and one time variables. The problems could be viewed as $\mathbb{R}^4$ analogs of Darboux problems in $\mathbb{R}^2$. In contrast to the planar Darboux problem the four-dimensional version is ill-posed, since its homogeneous adjoint problem has infinitely many classical solutions. Thus, in the framework of the classical solvability the problem is not Fredholm. Alternatively, it is known that for smooth right-hand side functions, there is a uniquely determined generalized solution that may have strong power-type singularity at one boundary point. This singularity is isolated at the vertex of the characteristic light cone and does not propagate along the cone. In this article we give a general existence result and find a priori estimates for singular solutions. A lengthy reference list is appended.

Keywords: wave equation, boundary value problems, generalized solution, singular solutions, propagation of singularities, special functions

1. Introduction

We study some boundary value problems for the wave equation in $\mathbb{R}^4$ that were proposed by Murrey Protter in the 1950s. Consider the wave equation with three space and one time variables

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - u_{tt} = f(x,t) \tag{1}$$

for $(x,t) = (x_1, x_2, x_3, t) \in \mathbb{R}^4$ in the domain

$$\Omega = \{(x,t) : 0 < t < 1/2,\ t < \sqrt{x_1^2 + x_2^2 + x_3^2} < 1 - t\}.$$

This domain is bounded by the two characteristic cones

$$\Sigma_1 = \{(x,t) : 0 < t < 1/2,\ \sqrt{x_1^2 + x_2^2 + x_3^2} = 1 - t\},$$

$$\Sigma_2 = \{(x,t) : 0 < t < 1/2,\ \sqrt{x_1^2 + x_2^2 + x_3^2} = t\}$$

and the ball

$$\Sigma_0 = \{t = 0,\ \sqrt{x_1^2 + x_2^2 + x_3^2} < 1\},$$

centered at the origin O; i.e., $x = 0$ and $t = 0$. The right-hand side function f of (1) satisfies some smoothness conditions in Ω that will be fixed later. We will study the following BVPs:

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Problem P1. *Find a solution of* (1) *in* Ω *satisfying the boundary conditions*

$$P1: \quad u|_{\Sigma_0} = 0, \quad u|_{\Sigma_1} = 0.$$

Problem P1*. *Find a solution of* (1) *in* Ω *satisfying the adjoint boundary conditions*

$$P1^*: \quad u|_{\Sigma_0} = 0, \quad u|_{\Sigma_2} = 0.$$

In this paper we give a general existence result and discuss the behavior of the generalized solution of Problem P1.

First, we present a brief historical overview here and provide an extensive list of references. Protter arrived at these problems while examining BVPs for mixed type equations which describe transonic flows in fluid dynamics. In particular, the classical two-dimensional Guderley–Morawetz problem for the Gellerstedt equation of hyperbolic-elliptic type which models flows around airfoils. The Guderley–Morawetz problem is well studied in the 1950s—see the surveys for the 2-D mixed type BVPs and the transonic models by Morawetz [1, 2]. For example, the existence of weak solutions and the uniqueness of strong solutions were proved by Morawetz [3], while Lax and Phillips [4] showed that the weak solutions are strong. In 1954 Protter [5, 6] formulated for $3D$ mixed-type equations (with two space and one time variables) some multidimensional analogs of the planar Guderley–Morawetz problem. The assumption was that the methods used to attack the $2D$ case could be applied for the multidimensional problems. However, the multidimensional case turns out to be rather different and the situation there is still unclear. Although, the uniqueness of the so-called quasiregular solutions is proved by Aziz and Schneider in [7], there are no general existence results for the Protter mixed-type problems. Even the question of well-posedness is not resolved completely.

As regards the results for existence or nonexistence of nontrivial solutions of related quasilinear problems of mixed hyperbolic-elliptic type in the multidimensional case, see [8, 9]. About results on BVPs for the multidimensional mixed-type Lavrent′ev–Bitsadze equation, see [10, 11].

In relation to the mixed-type problems, Protter also formulated and studied in [6] some BVPs in the hyperbolic part of the domain both for degenerated hyperbolic equations and for the wave equation—the $3D$ variants of Problems P1 and P1*. Later Paul Garabedian [12] gave the statement of such problems in $\mathbb{R}^4$ and proved the uniqueness of the classical solutions of Problem P1. Problems P1 and P1* in Ω could be considered as four-dimensional analogs of the planar Darboux problems (or the Cauchy–Goursat problems) for the string equations in a characteristic triangle. Initially, the expectation was that such multidimensional BVPs are classically solvable for very smooth right-hand side functions. Contrary to this traditional belief, soon it became clear that unlike the planar Darboux problem, the Protter problems are ill-posed. In fact, the homogeneous adjoint problem P1* has smooth classical solutions and the linear space they span is infinite-dimensional. Thus, in the frame of the classical solvability the Protter Problem P1 is not Fredholm, since it has infinite-dimensional cokernel. Alternatively, the notion of generalized solution that may have singularity on Σ_2 was introduced in [28]. In fact, it is known that the generalized solution has singularity isolated at only one point—the origin O. The point O lies both on the characteristic part of the boundary Σ_2 and on the noncharacteristic part Σ_0, and this case is different from the standard propagation of singularities (see Hörmander [13, Chapter 24.5]). A short survey and comparison of various recent results for Protter problems are in [14, 15]. In [16] the semi-Fredholm solvability of

Problem P1 is discussed. According to the classical and singular solutions let us mention here some results of Serik Aldashev in [10, 11, 17, 18] and a series of papers of Khe Kan Cher [19–22] and also the joint papers with his co-authors [23, 24].

Results for the wave equation but with lower order terms could be found in [25, 26]. Regarding results for degenerated hyperbolic equations we refer to [18, 27, 28], and for equations of Keldysh type—to [29]. Some other multidimensional analogs of the classical Darboux problem are considered in [30–33].

In the present article we give sufficient conditions on the right-hand function f for the existence of a generalized solution of Problem P1 and discuss its exact behavior. On the one hand, we need a priori estimates away from the origin to ensure the existence of the solution (like in Theorems 1 and 2). On the other hand, our goal is to study singularity near O. We find upper estimates for the growth the generalized solution at O in Theorem 2 and Corollary 1, and a lower estimate is given in Theorem 3.

2. Classical and Generalized Solutions

In order to construct the solutions of the homogeneous adjoint problem P1* we need in $\mathbb{R}^3$ the orthonormal system of the *spherical functions* Y_n^m ($n \in \mathbb{N} \cup \{0\}$, and $m = 1, \ldots, 2n+1$). They are defined usually on the unit sphere $S^2 := \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ in spherical polar coordinates (see [34]). Expressed in Cartesian coordinates here, we can define them by

$$Y_n^{2k}(x_1, x_2, x_3) = C_{n,k} \frac{d^k}{dx_3^k} P_n(x_3) \operatorname{Im}\{(x_1 + ix_2)^k\} \quad \text{for} \quad k = 1, \ldots, n; \tag{2}$$

$$Y_n^{2k+1}(x_1, x_2, x_3) = C_{n,k} \frac{d^k}{dx_3^k} P_n(x_3) \operatorname{Re}\{(x_1 + ix_2)^k\} \quad \text{for} \quad k = 0, \ldots, n,$$

where $C_{n,k}$ are constants and P_n are the *Legendre polynomials.* The Legendre polynomials are defined by the Rodrigues formula as

$$P_n(s) := \frac{1}{2^n n!} \frac{d^n}{ds^n} (s^2 - 1)^n = \sum_{k=0}^{[\frac{n}{2}]} a_{n,2k} s^{n-2k},$$

with the coefficients

$$a_{n,2k} = (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!}.$$

The constants $C_{n,m}$ are such that Y_n^m form a complete orthonormal system in $L_2(S^2)$ (see [34]). For convenience in the discussions that follow, we extend the spherical functions beyond S^2 radially, keeping the same notation Y_n^m for the extended functions, i.e., $Y_n^m(x) := Y_n^m(x/|x|)$ for $x \in \mathbb{R}^3 \backslash O$.

Let us define, for $k \in \mathbb{N} \cup \{0\}$, the functions

$$h_k(\xi, \eta) = \int_\eta^\xi s^k P_n\left(\frac{\xi\eta + s^2}{s(\xi + \eta)}\right) ds.$$

Then Lemmas 1.1 and 2.3 from [35] give the following solutions of the homogeneous adjoint problem.

Lemma 1 [35]. *The functions*

$$v_{k,m}^{n}(x,t) = |x|^{-1} h_{n-2k-2}\left(\frac{|x|+t}{2}, \frac{|x|-t}{2}\right) Y_n^m(x)$$

are classical solutions from $C^\infty(\Omega) \cap C(\overline{\Omega})$ of the homogeneous problem P1 for $n \in \mathbb{N}$, $m = 1, \ldots, 2n+1$ and $k = 0, 1, \ldots, [(n-1)/2] - 2$.*

Actually, the functions $v_{k,m}^n$ here are practically the same as the solutions from [35, Lemma 1.1]. On the other hand, [16, Theorem 1] suggests that there are no other linearly independent nontrivial classical solutions of the homogenous adjoint Problem P1*. Solutions for the homogenous adjoint problem were first found by Tong Kwang-Chang [39]. Some different representations of the solutions of the homogeneous problem P1* and the functions $v_{k,m}^n$ are given by Khe Kan Cher [22].

Naturally, a necessary condition for the existence of a classical solution for Problem P1 is the orthogonality of the right-hand side function f to all $v_{k,m}^n(x,t)$. To avoid infinitely many necessary conditions in the framework of the classical solvability, we introduce *generalized solutions* for Problem P1, eventually with singularity at the origin O.

Definition 1 [36]. A function $u = u(x,t)$ is called a *generalized solution* of Problem P1 in Ω, if the following are satisfied:

1) $u \in C^1(\overline{\Omega} \backslash O)$, $u|_{\Sigma_0 \backslash O} = 0, u|_{\Sigma_1} = 0$, and

2) we have

$$\int\limits_{\Omega} (u_t w_t - u_{x_1} w_{x_1} - u_{x_2} w_{x_2} - u_{x_3} w_{x_3} - f w)\, dx dt = 0 \tag{3}$$

for all $w \in C^1(\overline{\Omega})$ such that $w = 0$ on Σ_0 and a neighborhood of Σ_2.

Here we find some appropriate conditions for f under which there exists a generalized solution of Problem P1.

3. Existence of a Generalized Solution

The spherical functions form a complete orthonormal system in $L_2(S^2)$, and, generally, each smooth function $f(x,t)$ can be expanded as a harmonic series

$$f(x,t) = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} f_n^m(|x|, t) Y_n^m(x) \tag{4}$$

with the Fourier coefficients

$$f_n^m(r,t) := \int\limits_{S(r)} f(x,t) Y_n^m(x)\, d\sigma_r, \tag{5}$$

where $S(r)$ is the three-dimensional sphere in $x = (x_1, x_2, x_3)$ variables; i.e., $S(r) := \{x \in \mathbb{R}^3 : |x| = r\}$. In the previous paper [36], the Protter problem was studied in the special case when the right-hand side function is a finite Fourier sum, while in [16] for the general case $f \in C^1(\overline{\Omega})$ the necessary and sufficient conditions for the existence of bounded solutions were found. In fact, the behavior of the generalized solution depends strongly on the inner product (with respect to the $L_2(\Omega)$ inner product) of the right-hand side function $f(x,t)$ with the functions $v_{k,m}^n(x,t)$ from Lemma 1. Thus, let us denote by $\beta_{k,m}^n$ the parameters

$$\beta_{k,m}^n := \int\limits_{\Omega} v_{k,m}^n(x,t) f(x,t)\, dx dt, \tag{6}$$

where $n = 0, \ldots, l$, $k = 0, \ldots, \left[\frac{n-1}{2}\right]$ and $m = 1, \ldots, 2n+1$.

Theorem 1 [16]. *Let $f(x,t)$ belong to $C^{10}(\overline{\Omega})$. Then the necessary and sufficient conditions for existence of a bounded generalized solution $u(x,t)$ of Protter's Problem P1 are*

$$\int\limits_{\Omega} v^n_{k,m}(x,t)f(x,t)\,dxdt = 0 \tag{7}$$

for all $n \in \mathbb{N}$, $k = 0,\dots,\left[\frac{n-1}{2}\right]$, and $m = 1,\dots,2n+1$.

Moreover, this generalized solution $u(x,t) \in C^1(\overline{\Omega}\backslash O)$ satisfies the a priori estimates

$$|u(x,t)| \le C\|f\|_{C^{10}(\overline{\Omega})}, \tag{8}$$

$$\sum_{i=1}^{3} |u_{x_i}(x,t)| + |u_t(x,t)| \le C(|x|^2+t^2)^{-1}\|f\|_{C^{10}(\overline{\Omega})}, \tag{9}$$

where C is a constant independent of $f(x,t)$.

We turn now to investigation of the singular solution of Problem P1.

In order to formulate a general existence result, let us introduce, for $p \in \mathbb{R}$ and $k \in \mathbb{N}$, the series

$$\|f;n^p;C^k\| := \left\|f^0_0(|x|,t)\right\|_{C^0(\Omega)} + \sum_{n=1}^{\infty} n^p \left\|\sum_{m=1}^{2n+1} f^m_n(|x|,t)Y^m_n(x)\right\|_{C^k(\Omega)}$$

and the power series

$$\Phi(s) := \sum_{n=1}^{\infty}\left[\sum_{m=1}^{2n+1}\sum_{k=0}^{[n/2]}\left|\beta^n_{k,m}\right|\right]s^n.$$

Obviously, $\|f;n^{p_1};C^{k_1}\| \ge \|f;n^{p_2};C^{k_2}\|$ for $p_1 \ge p_2$ and $k_1 \ge k_2$.

Now we can formulate the main result in the present paper.

Theorem 2. *Let $f(x,t)$ belong to $C^1(\overline{\Omega})$. Suppose that the series $\|f;n^6;C^0\|$ and $\|f;n^4;C^1\|$ are convergent and the power series $\Phi(s)$ has an infinite radius of convergence. Then there exists a unique generalized solution $u(x,t) \in C^1(\overline{\Omega}\backslash O)$ of Protter's Problem P1 and it satisfies in $\overline{\Omega}\backslash O$ the a priori estimates*

$$|u(x,t)| \le C\left[\Phi\left(\frac{C_1}{|x|+t}\right) + |x|^{-1}\|f;n^4;C^0\|\right],$$

$$|u(x,t)| \le C\left[\Phi\left(\frac{C_1}{|x|+t}\right) + \|f;n^6;C^0\| + \|f;n^4;C^1\|\right],$$

$$\sum_{i=1}^{3}|u_{x_i}(x,t)| + |u_t(x,t)| \le C|x|^{-2}\left[\Phi\left(\frac{C_2}{|x|+t}\right) + \|f;n^6;C^0\|\right],$$

where C, C_1, and C_2 are constants independent of $f(x,t)$.

In these estimates, the singularity of the generalized solution at O is controlled by the function $\Phi(s)$, while $\|f;n^p;C^k\|$ bounds the "regular part" of $u(x,t)$.

Let us compare the situation here for the (3+1)-D case (three space and one time dimensions) with the results of [37] for (2+1)-D Protter Problems. According to Theorem 5.3 of [37], the sufficient condition for existence of generalized solution is the convergence of the series

$$\sum_{n=1}^{\infty}\frac{1}{n}I_0\left(\frac{2n}{\varepsilon}\right)\left(\left\|f^1_n\right\|_{C^0(\Omega)} + \left\|f^2_n\right\|_{C^0(\Omega)}\right) \quad \text{for all } \varepsilon > 0, \tag{10}$$

where f_n^i are the Fourier coefficients of the right-hand side which could be viewed as the analogs of the functions f_n^m given by (5). The function I_0 is the modified Bessel function of the first kind:

$$I_0(s) := \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{s}{2}\right)^{2k}.$$

Notice that we have the estimate

$$I_0(s) \le e^s \quad \text{for } s \ge 0.$$

Using the exponential function in (10) instead of I_0, we then get the following result somewhat weaker than [37, Theorem 5.3]. Suppose that the power series

$$\Phi_1(s) := \sum_{n=1}^{\infty} \left(\|f_n^1\|_{C^0(\Omega)} + \|f_n^2\|_{C^0(\Omega)}\right) n^{-1} s^n$$

converges for all s. Then for the singularity of the unique generalized solution $u(x,t)$ for the (2+1)-D Protter Problem, near the origin we have the estimate

$$|u(x,t)| \le C\Phi_1\left[\exp\left(\frac{2}{|x|}\right)\right]. \tag{11}$$

Here, in the (3+1)-D case, note that from the definition of $v_{k,m}^n$ in Lemma 1 it follows that

$$|v_{k,m}^n(x,t)| \le |Y_n^m(x)| \le C_1 n^{1/2}$$

and so

$$\left|\beta_{k,m}^n\right| \le C_2 n^{1/2} \left\|f_n^m\right\|_{C^0(\Omega)}.$$

Then, to compare with [37, Theorem 5.3] and (11), we can formulate the next result that ensues from Theorem 2.

Corollary 1. *Let $f(x,t)$ belong to $C^1(\overline{\Omega})$. Suppose that the series $\|f;n^6;C^0\|$ and $\|f;n^4;C^1\|$ are convergent and the power series*

$$\Phi_2(s) := \sum_{n=1}^{\infty}\left[\sum_{m=1}^{2n+1} \left\|f_n^m\right\|_{C^0(\Omega)}\right] s^n$$

has an infinite radius of convergence. Then the unique generalized solution $u(x,t) \in C^1(\overline{\Omega}\backslash O)$ of Protter's Problem P1 satisfies near the origin the estimate

$$|u(x,t)| \le C\Phi_2\left(\frac{C_0}{|x|+t}\right), \tag{12}$$

where C and C_0 are constants independent of $f(x,t)$.

4. Construction of Singular Solutions

In the special case when the right-hand side function f is a harmonic polynomial, the exact asymptotic formula for the generalized solution at O is found in [36]. It shows that the solution can have only power type singularity. However, in the general case $f(x,t) \in C^1(\overline{\Omega})$ some stronger singularities are also possible. Actually, in [38] the existence of generalized solutions with at least exponential growth at the origin is announced. The next theorem could be used to construct other singular solutions.

Theorem 3. *Let $f(x,t)$ belong to $C^1(\overline{\Omega})$, while the series $\|f; n^6; C^0\|$ and $\|f; n^4; C^1\|$ are convergent, and the power series $\Phi(s)$ has an infinite radius of convergence. Let the numbers $\alpha_p \geq 0$, $p = 0, 1, 2, \ldots$, be such that the series*

$$\phi(s) := \sum_{p=0}^{\infty} \alpha_p s^p$$

converges for all $s \in \mathbb{R}$. Suppose that there is $x^* = (x_1^*, x_2^*, x_3^*) \in \mathbb{R}^3$ such that

$$\sum_{k=0}^{\infty} \sum_{m=1}^{2p+4k+1} p a_{2k} \beta_{m,k}^{p+2k} Y_{p+2k}^m(x^*) \geq \alpha_p. \tag{13}$$

Then there exists a number $\delta \in (0, 1/2)$ that the unique generalized solution $u(x,t)$ of Problem P1 satisfies the estimate

$$|u(tx_1^*, tx_2^*, tx_3^*, t)| \geq \phi\left(\frac{1}{2t}\right)$$

for $t \in (0, \delta)$.

Remark. We can find a right-hand side $f(x,t) \in C^1(\overline{\Omega})$ by choosing suitable Fourier coefficients $f_n^m(r,t)$ "small enough" so that the required series $\|f; n^p 4; C^k\|$ and $\Phi(s)$ be convergent. At the same time, selecting the functions f_n^m that satisfy (13) with larger constants α_p will produce solutions with a stronger singularity. In accordance with the result from [38], it is possible to obtain an appropriate function f with constants $\alpha_p = (p!)^{-1}$ for all p, and so the corresponding solution has exponential growth at O.

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Nedyu Popivanov; Todor Popov
Department of Mathematics and Informatics,
University of Sofia, 1164 Sofia, Bulgaria
`nedyu@fmi.uni-sofia.bg; topopover@fmi.uni-sofia.bg`

Rudolf Scherer
Institute for Applied and Numerical Mathematics,
Karlsruhe Institute of Technology (KIT),
76128 Karlsruhe, Germany
`rudolf.scherer@kit.edu`

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STABILITY ESTIMATES AND APPROXIMATE SOLUTIONS TO A BOUNDARY VALUE PROBLEM FOR A FORTH ORDER PARTIAL DIFFERENTIAL EQUATION

K. S. Fayazov and I. O. Khajiev

Abstract. Under study is some initial-boundary value problem for a forth order equation of mixed type. The problem belongs to the class of strongly ill-posed problems of mathematical physics. In accord with A. N. Tikhonov's ideas, we establish conditional well-posedness of this problem. Involving spectral decompositions and energy integrals, we prove uniqueness of a solution and its conditional stability on the well-posedness set. An approximate solution is constructed by the regularization method, and an estimate of the norm of the difference between the exact and approximate solutions is obtained. The regularization parameter is calculated from the minimality condition for an estimate of the norm of the difference between the exact and regularized solutions.

Keywords: mixed type equation, ill-posed problem, a priori estimate, uniqueness, stability, well-posedness set, regularization

Consider the equation

$$\operatorname{sgn} x \frac{\partial^4 u(x,t)}{\partial t^4} + \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \tag{1}$$

in the domain $\Omega = \{-1 < x < 1,\ x \neq 0,\ 0 < t < T\}$.

Problem. *Find a function $u(x,t)$ satisfying* (1) *in Ω, the initial conditions*

$$\left.\frac{\partial^j u(x,t)}{\partial t^j}\right|_{t=0} = \varphi_j(x), \quad j = 0,1,2,3,\ -1 \leq x \leq 1, \tag{2}$$

the boundary conditions

$$u(-1,t) = u(1,t) = 0, \quad 0 \leq t \leq T, \tag{3}$$

and the gluing conditions

$$u(-0,t) = u(+0,t), \quad u_x(-0,t) = u_x(+0,t), \quad 0 \leq t \leq T. \tag{4}$$

The ill-posed boundary value problems for model differential equations were examined by many authors, in particular, by Carleman, Hörmander, Nirenberg, Lavrent′ev, Landis, Jon, Levin, Kreĭn, et al.; and for operator-differential equations, by Kreĭn, Levin, Fayazov, and some other authors.

The problem (1)–(4) is Hadamard ill-posed. In this article we study the conditional well-posedness of (1)–(4) and construct an approximate solution stable under changes of data on the well-posedness set. In Section 1 an a priori estimate for a solution is derived, and the uniqueness of a solution and conditional stability of (1)–(4)

are proven in Section 2. In Section 3 the regularization method is applied to construct an approximate solution and an estimate of the difference between the exact and approximate solutions is obtained. A formula for the regularization parameter is derived from the optimality condition for the estimate.

Gevrey's articles were the first devoted to forward-backward parabolic equations. The solvability theory of boundary value problems for these equations was developed in the articles by Tersenov, Nakhushev, Egorov, Kislov, Pyatkov, Popov, and many other authors. The mixed type equations are studied in the articles by Bitsadze, Salakhitdinov, Vragov, Kozhanov, Pyatkov, et al. (see [1, 2] and the bibliography therein). The ill-posed problems are treated in [3–5].

1. An A Priori Estimate

In what follows, we use properties of eigenfunctions of the spectral problem

$$\begin{cases} \operatorname{sgn} x X''(x) + \lambda X(x) = 0, \\ X(-1) = X(1) = 0, \\ X(-0) = X(+0),\ X'(-0) = X'(+0). \end{cases} \tag{5}$$

Let $\{X_k^+\}_{k=1}^\infty$ and $\{X_k^-\}_{k=1}^\infty$ be the eigenfunctions of (5) with positive and negative eigenvalues λ_k^+ and λ_k^- such that the sequences λ_k^+ and $-\lambda_k^-$ are nondecreasing.

Note that

$$X_k^\pm(x) = \begin{cases} \sinh\sqrt{\lambda_k^\pm}\sin\sqrt{\lambda_k^\pm}(1-x), & 0 \le x \le 1, \\ \sin\sqrt{\lambda_k^\pm}\sinh\sqrt{\lambda_k^\pm}(1+x), & 0 \le x \le 1, \end{cases}$$

where $\pm\lambda_k^\pm = \mu_k^2$, $k = 1, 2, \dots$, with μ_k a positive root of the equation $\tan\mu = -\tanh\mu$.

Denote by $(u, v) = \int_{-1}^1 uv\,dx$ the inner product on $L_2(-1, 1)$, $\|u\|^2 = (u, u)$,

$$(\operatorname{sgn} x X_k^\pm, X_j^\pm) = \delta_{kj}, \quad \delta_{kj} = \begin{cases} 1, & k = j, \\ 0, & k \ne j. \end{cases}$$

Let $P^\pm$ be the spectral projections defined by the equalities

$$P^\pm u = \sum_{k=1}^\infty (\operatorname{sgn} x u, X_k^\pm) X_i^\pm.$$

In accord with [2], we have

$$(P^+ - P^-)u = u, \quad (\operatorname{sgn} x(P^+ - P^-)u, u) = \|u\|_0^2,$$

$$(\operatorname{sgn} x P^\pm u, v) = (\operatorname{sgn} x u, P^\pm v), \quad u, v \in H_0 = L_2(-1, 1),$$

$$\|u(x,t)\|_0^2 = \sum_{k=1}^\infty \{|(\operatorname{sgn} x u(x,t), X_k^+)|^2 + |(\operatorname{sgn} x u(x,t), X_k^-)|^2\}. \tag{6}$$

In accord with the results of [2], the eigenfunctions of (5) form a Riesz basis for H_0 and the norm (6) in $L_2(-1; 1)$ is equivalent to the initial norm.

By a *generalized solution to the boundary value problem* (1)–(4) we mean a function $u(x,t)$ such that $u(x,t) \in (C[0, T]; L_2(-1, 1))$,

$$\int_0^T \int_{-1}^1 u(x,t)(\mathrm{sgn}\, x\, V_{tttt} + V_{xx})\, dxdt$$

$$= \int_0^T \int_{-1}^1 \mathrm{sgn}\, x(\varphi_3(x)V(x,0) - \varphi_2(x)V_t(x,0) + \varphi_1(x)V_{tt}(x,0) - \varphi_0(x)V_{ttt}(x,0))\, dxdt$$

for all $V(x,t) \in W_2^4(\Omega)$, $V(x,T) = V_t(x,T) = V_{tt}(x,T) = V_{ttt}(x,T) = 0$, and $V(-1,t) = V(1,t) = 0$.

Assume that a solution to (1)–(4) exists and is representable as

$$u(x,t) = \sum_{k=1}^{\infty} u_k^+(t)X_k^+ + \sum_{k=1}^{\infty} u_k^-(t)X_k^-,$$

where

$$u_k^{\pm}(t) = \int_{-1}^1 \mathrm{sgn}\, xu(x,t)X_k^{\pm}(x)\, dx, \quad k = 1, 2, 3, \ldots.$$

As is easily seen, the functions $u_k^{\pm}(t)$ for all $k = 1, 2, 3, \ldots$ solve the problems

$$\begin{cases} \{u_k^+(t)\}_{tttt} - \mu_k^2 u_k^+(t) = 0, \\ u_k^+(0) = \varphi_{0_k}^+, \ \{u_k^+(0)\}_t = \varphi_{1_k}^+, \ \{u_k^+(0)\}_{tt} = \varphi_{2_k}^+, \ \{u_k^+(0)\}_{ttt} = \varphi_{3_k}^+, \end{cases} \tag{7}$$

$$\begin{cases} \{u_k^-(t)\}_{tttt} + \mu_k^2 u_k^-(t) = 0, \\ u_k^-(0) = \varphi_{0_k}^-, \ \{u_k^-(0)\}_t = \varphi_{1_k}^-, \ \{u_k^-(0)\}_{tt} = \varphi_{2_k}^-, \ \{u_k^-(0)\}_{ttt} = \varphi_{3_k}^-, \end{cases} \tag{8}$$

where

$$\varphi_{j_k}^{\pm} = \int_{-1}^1 \mathrm{sgn}\, x\varphi_j(x)X_k^{\pm}(x)\, dx, \quad j = 0, 1, 2, \ldots.$$

Lemma 1 [6]. *A solution to the equation $\phi''(t) - \theta\phi(t) = 0$ $(0 < t < T)$ satisfying the conditions $\phi(0) = p$ and $\phi'(0) = q$ meets the estimate*

$$\phi^2(t) \le (p^2 + |\delta|)^{\frac{T-t}{T}}(\phi^2(T) + |\delta|)^{\frac{t}{T}} e^{2t(T-t)} - |\delta|,$$

where θ is a constant and $\delta = \frac{1}{2}(\theta p^2 - q^2)$.

Consider (7) and introduce the notations

$$\frac{1}{\mu_k}\frac{d^2u_k^+}{dt^2} = \vartheta_k^+, \quad w_k^+ = u_k^+ - \vartheta_k^+, \quad v_k^+ = u_k^+ + \vartheta_k^+.$$

Some transformations lead to the following:

$$\{v_k^+\}_{tt} - \mu_k v_k^+ = 0, \quad v_k^+(0) = \varphi_{0_k}^+ + \mu_k^{-1}\varphi_{2_k}^+, \quad \{v_k^+(0)\}_t = \varphi_{1_k}^+ + \mu_k^{-1}\varphi_{3_k}^+, \tag{9}$$

$$\{w_k^+\}_{tt} + \mu_k w_k^+ = 0, \quad w_k^+(0) = \varphi_{0_k}^+ - \mu_k^{-1}\varphi_{2_k}^+, \quad \{w_k^+(0)\}_t = \varphi_{1_k}^+ - \mu_k^{-1}\varphi_{3_k}^+. \tag{10}$$

By Lemma 1, solutions to (9), (10) satisfy the estimates

$$\{v_k^+(t)\}^2 \le (\{v_k^+(0)\}^2 + |\alpha_k^+|)^{\frac{T-t}{T}}(\{v_k^+(T)\}^2 + |\alpha_k^+|)^{\frac{t}{T}} e^{2t(T-t)} - |\alpha_k^+|,$$

$$\{w_k^+(t)\}^2 \le (\{w_k^+(0)\}^2 + |\beta_k^+|)^{\frac{T-t}{T}}(\{w_k^+(T)\}^2 + |\beta_k^+|)^{\frac{t}{T}} e^{2t(T-t)} - |\beta_k^+|,$$

where $\alpha_k^+ = 0.5\big(\mu_k\{v_k^+(0)\}^2 - \{v_k^+(0)\}_t^2\big)$ and $\beta_k^+ = 0.5\big(\mu_k\{w_k^+(0)\}^2 + \{w_k^+(0)\}_t^2\big)$.

Note that $u_k^+ = 0.5(v_k^+ + w_k^+)$ and $\{u_k^+\}^2 \le 0.5(\{v_k^+\}^2 + \{w_k^+\}^2)$. In this case

$$\{u_k^+\}^2 \le \frac{e^{2t(T-t)}}{2}\Bigg((\{v_k^+(0)\}^2 + |\alpha_k^+|)^{\frac{T-t}{T}}\Bigg(\{u_k^+(T)\}^2 + \frac{1}{\mu_k^2}\{u_k^+(T)\}_{tt}^2 + |\alpha_k^+|\Bigg)^{\frac{t}{T}}$$
$$+(\{w_k^+(0)\}^2 + |\beta_k^+|)^{\frac{T-t}{T}}\Bigg(\{u_k^+(T)\}^2 + \frac{1}{\mu_k^2}\{u_k^+(T)\}_{tt}^2 + |\beta_k^+|\Bigg)^{\frac{t}{T}}\Bigg). \tag{11}$$

Proceed with an estimate for a solution to (8). Consider the equation

$$\frac{d^4h}{dt^4} = -\mu^2 h \tag{12}$$

for $0 < t < T$ with the conditions

$$\frac{d^j h}{dt^j}\bigg|_{t=0} = f_j, \quad j = 0, 1, 2, \dots, \tag{13}$$

where μ is a constant. Rewrite (12) as

$$(\partial_t - r_1)(\partial_t - r_2)(\partial_t - r_3)(\partial_t - r_4)h = 0, \tag{14}$$

where

$$r_1 = l + il, \quad r_2 = -l + il, \quad r_3 = -l - il, \quad r_4 = l - il, \quad l = \frac{\sqrt{2\mu}}{2},$$

and r_j, $j = 1, 2, 3, 4$, are the roots of the equation $r^4 + \mu^2 = 0$.

Equation (14) can be rewritten as the system

$$\begin{cases} (\partial_t - r_4)h = v, \\ (\partial_t - r_3)v = w, \\ (\partial_t - r_2)w = z, \\ (\partial_t - r_1)z = 0. \end{cases} \tag{15}$$

Let $z(t)$ be a solution to the equation

$$z_t - r_1 z = 0. \tag{16}$$

Obviously, it is representable as $z(t) = x(t) + iy(t)$, where $x(t)$ and $y(t)$ are solutions to the system

$$\begin{cases} x_t = l(x - y), \\ y_t = l(x + y). \end{cases}$$

As is easily seen, $z(t)$ satisfies the estimate

$$|z(t)|^2 \le (|z(0)|^2)^{1-\frac{t}{T}}(|z(T)|^2)^{\frac{t}{T}},$$

where $|z(t)|^2 = x^2(t) + y^2(t)$. In this case (14) yields $|z(t)|^2 \le \gamma_1(t)$, with

$$\gamma_1(t) = (\mu^3|f_0|^2 + \mu^2|f_1|^2 + \mu|f_2|^2 + |f_3|^2)^{1-\frac{t}{T}}$$
$$\times(\mu^3|h(T)|^2 + \mu^2|h_t(T)|^2 + \mu|h_{tt}(T)|^2 + |h_{ttt}(T)|^2)^{\frac{t}{T}}.$$

Let $w(t)$ be a solution to the nonhomogeneous equation

$$w_t - k_2 w = z, \tag{17}$$

which is representable as

$$w(t) = \omega(t) + \widetilde{\omega}(t),$$

where $\omega(t)$ is a general solution of the homogeneous equation and $\widetilde{\omega}(t)$ is a particular solution to (17). Since $\widetilde{\omega}(t) = \tilde{x}(t) + i\tilde{y}(t)$, we have

$$\begin{cases} \tilde{x}_t = -l(\tilde{x} + \tilde{y}) + x, \\ \tilde{y}_t = l(\tilde{x} - \tilde{y}) + y. \end{cases} \tag{18}$$

Obviously, a particular solution to (18) is representable as

$$\tilde{x}(t) = \frac{\cos lt}{2} \int_0^t (x(\tau)\sin l\tau + y(\tau)\cos l\tau) e^{l(\tau - t)}\, d\tau$$

$$+ \frac{\sin lt}{2} \int_0^t (x(\tau)\sin l\tau - y(\tau)\cos l\tau) e^{l(\tau - t)}\, d\tau,$$

$$\tilde{y}(t) = \frac{\sin lt}{2} \int_0^t (x(\tau)\sin l\tau + y(\tau)\cos l\tau) e^{l(\tau - t)}\, d\tau$$

$$- \frac{\cos lt}{2} \int_0^t (x(\tau)\sin l\tau - y(\tau)\cos l\tau) e^{l(\tau - t)}\, d\tau.$$

We conclude that

$$|\tilde{x}(t)| \le \int_0^t (|x(\tau)| + |y(\tau)|)\, d\tau, \quad |\tilde{y}(t)| \le \int_0^t (|x(\tau)| + |y(\tau)|)\, d\tau$$

or

$$\tilde{x}(t)^2 \le 2T \int_0^t \left(x(\tau)^2 + y(\tau)^2\right) d\tau, \quad \tilde{y}(t)^2 \le 2T \int_0^t \left(x(\tau)^2 + y(\tau)^2\right) d\tau;$$

in this case

$$\tilde{x}(t)^2 + \tilde{y}(t)^2 \le 4T \int_0^t \left(x(\tau)^2 + y(\tau)^2\right) d\tau,$$

i.e., $|\widetilde{\omega}(t)|^2 = \tilde{x}(t)^2 + \tilde{y}(t)^2 \le 4T \int_0^T |z(t)|^2\, dt$.

The function $\omega(t)$ satisfies the estimate

$$|\omega(t)|^2 \le (|\omega(0)|^2)^{1 - \frac{t}{T}} (|\omega(T)|^2)^{\frac{t}{T}}.$$

Hence,

$$|w(t)|^2 \le (|w(0)|^2)^{1 - \frac{t}{T}} (|w(T)|^2 + |\widetilde{\omega}(T)|^2)^{\frac{t}{T}} + |\widetilde{\omega}(T)|^2.$$

From (15) it follows that $|w(t)|^2 \le \gamma_2(t)$, where

$$\gamma_2(t) = (\mu^2 |f_0|^2 + \mu |f_1|^2 + |f_2|^2)^{1 - \frac{t}{T}}$$

$$\times \left(\mu^2 |h(T)|^2 + \mu |h_t(T)|^2 + |h_{tt}(T)|^2 + 2T \int_0^T \gamma_1(t)\, dt \right)^{\frac{t}{T}} + 4T \int_0^T \gamma_1(t)\, dt.$$

Similar arguments for solutions to the equations $v_t - r_3 v = w$ and $h_t - r_4 h = v$ validate the estimates

$$\begin{aligned} |v(t)|^2 &\le (|v(0)|^2)^{1-\frac{t}{T}}(|v(T)|^2 + |\tilde{v}(T)|^2)^{\frac{t}{T}} + |\tilde{v}(T)|^2, \\ |h(t)|^2 &\le (|h(0)|^2)^{1-\frac{t}{T}}(|h(T)|^2 + |\tilde{h}(T)|^2)^{\frac{t}{T}} + |\tilde{h}(T)|^2, \end{aligned} \tag{19}$$

where

$$|\tilde{v}(T)|^2 \le 4T \int_0^T |w(t)|^2\,dt, \quad |\tilde{h}(T)|^2 \le 4T \int_0^T |v(t)|^2\,dt.$$

From the above we infer

$$|v(t)|^2 \le \gamma_3(t),$$

$$|h(t)|^2 \le (|f_0|^2)^{1-\frac{t}{T}}\left(|h(T)|^2 + 4T\int_0^T \gamma_3(t)\,dt\right)^{\frac{t}{T}} + 4T\int_0^T \gamma_3(t)\,dt,$$

where

$$\gamma_3(t) = (\mu|f_0|^2 + |f_1|^2)^{1-\frac{t}{T}}\left(\mu|h(T)|^2 + |h_t(T)|^2 + 2T\int_0^T \gamma_2(t)\,dt\right)^{\frac{t}{T}} + 4T\int_0^T \gamma_2(t)\,dt(t).$$

Thus, a solution to (8) satisfies the estimate

$$|u_k^-(t)|^2 \le (|\varphi_{0_k}^-|^2)^{1-\frac{t}{T}}\left(|u_k^-(T)|^2 + 4T\int_0^T \gamma_{3_k}(t)\,dt\right)^{\frac{t}{T}} + 4T\int_0^T \gamma_{3_k}(t)\,dt, \tag{20}$$

where

$$\gamma_{3_k}(t) = (\mu_k|\varphi_{0_k}^-|^2 + |\varphi_{1_k}^-|^2)^{1-\frac{t}{T}}$$

$$\times\left(\mu_k|u_k^-(T)|^2 + |\{u_k^-(T)\}_t|^2 + 2T\int_0^T \gamma_{2_k}(t)\,dt\right)^{\frac{t}{T}} + 4T\int_0^T \gamma_{2_k}(t)\,dt,$$

$$\gamma_{2_k}(t) = \left(\mu_k^2|\varphi_{0_k}^-|^2 + \mu_k|\varphi_{1_k}^-|^2 + |\varphi_{2_k}^-|^2\right)^{1-\frac{t}{T}}$$

$$\times\left(\mu_k^2|u_k^-(T)|^2 + \mu_k|\{u_k^-(T)\}_t|^2 + |\{u_k^-(T)\}_{tt}|^2 + 2T\int_0^T \gamma_{1_k}(t)\,dt\right)^{\frac{t}{T}} + 4T\int_0^T \gamma_{1_k}(t)\,dt,$$

$$\gamma_{1_k}(t) = \left(\mu_k^3|\varphi_{0_k}^-|^2 + \mu_k^2|\varphi_{1_k}^-|^2 + \mu_k|\varphi_{2_k}^-|^2 + |\varphi_{3_k}^-|^2\right)^{1-\frac{t}{T}}$$

$$\times\left(\mu_k^3|u_k^-(T)|^2 + \mu_k^2|\{u_k^-(T)\}_t|^2 + \mu_k|\{u_k^-(T)\}_{tt}|^2 + |\{u_k^-(T)\}_{ttt}|^2\right)^{\frac{t}{T}}.$$

Here $k = 1, 2, \dots$.

2. Uniqueness and Conditional Stability Theorems

Introduce the notations

$$M = \left\{u : \sum_{k=1}^{\infty}\left(\sum_{j=0}^{3}\mu_k^{3-j}\left(\frac{d^j u_k^+(T)}{dt^j}\right)^2 + \mu_k^{3-j}\left(\frac{d^j u_k^-(T)}{dt^j}\right)^2\right) \le m^2\right\},$$

and put

$$\|\varphi_0(x)\|_3 = \sum_{k=1}^{\infty}\mu_k^3(\{\varphi_{0_k}^+\}^2 + \{\varphi_{0_k}^-\}^2), \quad \|\varphi_1(x)\|_2 = \sum_{k=1}^{\infty}\mu_k^2(\{\varphi_{1_k}^+\}^2 + \{\varphi_{1_k}^-\}^2), \tag{21}$$

$$\|\varphi_2(x)\|_1 = \sum_{k=1}^{\infty}\mu_k(\{\varphi_{2_k}^+\}^2 + \{\varphi_{2_k}^-\}^2), \quad \|\varphi_3(x)\|_0 = \sum_{k=1}^{\infty}(\{\varphi_{3_k}^+\}^2 + \{\varphi_{3_k}^-\}^2). \tag{22}$$

Theorem 1. *Let a solution $u(x,t) \in M$ to (1)–(4) exist. Then it is unique.*

Proof. Let $u_1(x,t)$, $u_2(x,t)$ be two solutions to (1)–(4) with the same data. The function $u(x,t) = u_1(x,t) - u_2(x,t)$ is a solution to (1)–(4) with the zero data. Since

$$\|u(x,t)\|_0^2 = \sum_{k=1}^{\infty} |u_k^+|^2 + \sum_{k=1}^{\infty} |u_k^-|^2,$$

(11) and (20) imply that $\sum_{k=1}^{\infty} |u_k^-|^2 + \sum_{k=1}^{\infty} |u_k^+|^2 \le 0$. Therefore, $\|u(x,t)\|_0 \le 0$ and so $u(x,t) = 0$, or $u_1(x,t) = u_2(x,t)$ for all $(x,t) \in \Omega$.

Theorem 2. *Let a solution $u(x,t) \in M$ to (1)–(4) exist. Assume that*

$$\|\varphi_0(x) - \varphi_{0_\varepsilon}(x)\|_3 \le \varepsilon, \quad \|\varphi_1(x) - \varphi_{1_\varepsilon}(x)\|_2 \le \varepsilon,$$
$$\|\varphi_2(x) - \varphi_{2_\varepsilon}(x)\|_1 \le \varepsilon, \quad \|\varphi_3(x) - \varphi_{3_\varepsilon}(x)\|_0 \le \varepsilon.$$

Then every solution to (1)–(4) for $(x,t) \in \Omega$ satisfies the inequality

$$\|u(x,t)\|_0 \le \varpi(\varepsilon, m),$$

where $\varpi(\varepsilon,m) = \inf_t \{ (2(8\varepsilon^2)^{\frac{T-t}{T}} (m^2+s)^{\frac{t}{T}} e^{2t(T-t)} + s)^{1/2} \}$.

Proof. Assume that a solution to (1)–(4) exists, while (21) and (22) hold. Consider the difference

$$U(x,t) = u(x,t) - u_\varepsilon(x,t),$$

with $u(x,t)$ and $u_\varepsilon(x,t)$ solutions to (1)–(4) with the exact and approximate data, respectively. The function $U(x,t)$ is a solution to the equation

$$\operatorname{sgn} x \frac{\partial^4 U(x,t)}{\partial t^4} + \frac{\partial^2 U(x,t)}{\partial x^2} = 0$$

in Ω satisfying the initial conditions

$$U(x,0) = \varphi_0(x) - \varphi_{0_\varepsilon}(x), \quad U_t(x,0) = \varphi_1(x) - \varphi_{1_\varepsilon}(x),$$
$$U_{tt}(x,0) = \varphi_2(x) - \varphi_{2_\varepsilon}(x), \quad U_{ttt}(x,0) = \varphi_3(x) - \varphi_{3\varepsilon}(x), \quad -1 \le x \le 1,$$

the boundary conditions

$$U(-1,t) = U(1,t) = 0, \quad 0 \le t \le T,$$

and the gluing conditions

$$U(-0,t) = U(+0,t), \quad U_x(-0,t) = U_x(+0,t), \quad 0 \le t \le T.$$

A solution to this problem satisfies the inequality

$$\sum_{k=1}^{\infty} \{U_k^+\}^2 \le \frac{e^{2t(T-t)}}{2} \sum_{k=1}^{\infty} \Big((\{v_{k\varepsilon}^+(0)\}^2 + |\alpha_k^+|)^{\frac{T-t}{T}} \big(\{U_k^+(T)\}^2 + \mu_k^{-2}\{U_k^+(T)\}_{tt}^2 + |\alpha_{k\varepsilon}^+|\big)^{\frac{t}{T}}$$
$$+ (\{\omega_{k_\varepsilon}^+(0)\}^2 + |\beta_{k_\varepsilon}^+|)^{\frac{T-t}{T}} \big(\{U_k^+(T)\}^2 + \mu_k^{-2}\{U_k^+(T)\}_{tt}^2 + |\beta_{k_\varepsilon}^+|\big)^{\frac{t}{T}} \Big),$$

where

$$\sum_{k=1}^{\infty} \{v_{k\varepsilon}^+(0)\}^2 = \sum_{k=1}^{\infty} \big(\varphi_{0_k}^+ - \varphi_{0_{k\varepsilon}}^+ + \mu_k^{-1}(\varphi_{2_k}^+ - \varphi_{2_{k\varepsilon}}^+)\big)^2 \le 4\varepsilon^2,$$

$$\sum_{k=1}^{\infty} |\alpha_{k\varepsilon}^+| = 0,5 \sum_{k=1}^{\infty} \big|\mu_k \{v_{k\varepsilon}^+(0)\}^2 - \{v_{k\varepsilon}^+(0)\}_t^2\big| \le 4\varepsilon^2,$$

$$\sum_{k=1}^{\infty} \{\omega_{k\varepsilon}^+(0)\}^2 \le 4\varepsilon^2, \quad \sum_{k=1}^{\infty} |\beta_k^+|^2 \le 4\varepsilon^2, \quad \sum_{k=1}^{\infty} \big(\{U_k^+(T)\}^2 + \mu_k^{-2}\{U_k^+(T)\}_{tt}^2\big) \le m^2.$$

Simple transformations lead to the inequality

$$\sum_{k=1}^{\infty} \{U_k^+\}^2 \leq (8\varepsilon^2)^{\frac{T-t}{T}}(m^2+4\varepsilon^2)^{\frac{t}{T}} e^{2t(T-t)}.$$

Similarly, the expression $\sum_{k=1}^{\infty} \{U_k^-\}^2$ is estimated as

$$\sum_{k=1}^{\infty} |U_k^-(t)|^2 \leq (\varepsilon^2)^{1-\frac{t}{T}} \left(m^2 + 4T\int_0^T \gamma_{3_{k\varepsilon}}(m,t)\,dt\right)^{\frac{t}{T}} + 4T\int_0^T \gamma_{3_{k\varepsilon}}(m,t)\,dt,$$

where

$$\gamma_{3_{k\varepsilon}}(m,t) = (2\varepsilon^2)^{1-\frac{t}{T}} \left(m^2 + 2T\int_0^T \gamma_{2_{k\varepsilon}}(m,t)\,dt\right)^{\frac{t}{T}} + 4T\int_0^T \gamma_{2_{k\varepsilon}}(m,t)\,dt,$$

$$\gamma_{2_{k\varepsilon}}(m,t) = (3\varepsilon^2)^{1-\frac{t}{T}} \left(m^2 + 2T\int_0^T \gamma_{1_{k\varepsilon}}(m,t)\,dt\right)^{\frac{t}{T}} + 4T\int_0^T \gamma_{1_{k\varepsilon}}(m,t)\,dt,$$

$$\gamma_{1_{k\varepsilon}}(m,t) = (4\varepsilon^2)^{1-\frac{t}{T}}(m^2)^{\frac{t}{T}}.$$

Let

$$s = \max\left\{4\varepsilon^2, 4T\int_0^T \gamma_{3_{k\varepsilon}}(m,t)\,dt\right\}.$$

In this case

$$\|U(x,t)\|_0^2 \leq 2(8\varepsilon^2)^{\frac{T-t}{T}}(m^2+s)^{\frac{t}{T}} e^{2t(T-t)} + s.$$

Hence,

$$\|U(x,t)\| \leq \varpi(\varepsilon, m),$$

where

$$\varpi(\varepsilon,m) = \inf_t \{(2(8\varepsilon^2)^{\frac{T-t}{T}}(m^2+s)^{\frac{t}{T}} e^{2t(T-t)} + s)^{1/2}\}.$$

3. Approximate Solutions

Without loss of generality, we assume that $\varphi_1(x) = 0$, $\varphi_2(x) = 0$, and $\varphi_3(x) = 0$. A solution to (1)–(4), if existent, is representable as

$$u(x,t) = \sum_{k=1}^{\infty}\left(\frac{\varphi_{0_k}^+}{2}\cosh(\sqrt{\mu_k}t) + \frac{\varphi_{0_k}^+}{2}\cos(\sqrt{\mu_k}t)\right)X_k^+(x)$$
$$+\sum_{k=1}^{\infty}\left(\varphi_{0_k}^-\cosh\left(\sqrt{\frac{\mu_k}{2}}t\right)\cos\left(\sqrt{\frac{\mu_k}{2}}t\right)\right)X_k^-(x),$$

where

$$\varphi_{0_k}^{\pm} = \int_{-1}^{1} \operatorname{sgn} x\,\varphi_0(x) X_k^{\pm}(x)\,dx, \quad k = 1,2,\ldots.$$

The approximate solution is defined as

$$u^N(x,t) = \sum_{k=1}^{N} \left(\frac{\varphi_{0_k}^+}{2} \cosh(\sqrt{\mu_k} t) + \frac{\varphi_{0_k}^+}{2} \cos(\sqrt{\mu_k} t) \right) X_k^+(x)$$
$$+ \sum_{k=1}^{\infty} \left(\varphi_{0_k}^- \cosh\left(\sqrt{\frac{\mu_k}{2}} t \right) \cos\left(\sqrt{\frac{\mu_k}{2}} t \right) \right) X_k^-(x),$$

where N is an integer parameter of regularization, and an approximate solution with approximate data as

$$u^{N_\varepsilon}(x,t) = \sum_{k=1}^{N} \left(\frac{\varphi_{0_{k\varepsilon}}^+}{2} \cosh(\sqrt{\mu_k} t) + \frac{\varphi_{0_{k\varepsilon}}^+}{2} \cos(\sqrt{\mu_k} t) \right) X_k^+(x)$$
$$+ \sum_{k=1}^{\infty} \left(\varphi_{0_{k\varepsilon}}^- \cosh\left(\sqrt{\frac{\mu_k}{2}} t \right) \cos\left(\sqrt{\frac{\mu_k}{2}} t \right) \right) X_k^-(x),$$

where

$$\varphi_{0_{k\varepsilon}}^{\pm} = \int_{-1}^{1} \operatorname{sgn} x \varphi_{0_\varepsilon}(x)\, X_k^{\pm}(x)\, dx, \quad k = 1, 2, \dots .$$

Let $\|\varphi_0(x) - \varphi_{0_\varepsilon}(x)\|_3 \le \varepsilon$ and $u \in M$. Estimate the difference

$$\|u - u^{N_\varepsilon}\|_0 \le \|u - u^N\|_0 + \|u^N - u^{N_\varepsilon}\|_0. \tag{23}$$

The second term on the right-hand side of (23) is estimated as

$$\|u^N - u^{N_\varepsilon}\|_0^2 = \sum_{k=1}^{N} \left(\frac{1}{2}(\varphi_{0_k}^+ - \varphi_{0_{k\varepsilon}}^+) \cosh(\sqrt{\mu_k} t) + \frac{1}{2}(\varphi_{0_k}^+ - \varphi_{0_{k\varepsilon}}^+) \cos(\sqrt{\mu_k} t) \right)^2$$
$$+ \sum_{k=1}^{N} \left((\varphi_k^- - \varphi_{k_\varepsilon}^-) \cosh\left(\sqrt{\frac{\mu_k}{2}} t \right) \cos\left(\sqrt{\frac{\mu_k}{2}} t \right) \right)^2$$
$$\le \sum_{k=1}^{N} \cosh^2(\sqrt{\mu_k} t)((\varphi_{0_k}^+ - \varphi_{0_{k\varepsilon}}^+)^2 + (\varphi_{0_k}^- - \varphi_{0_{k\varepsilon}}^-)^2) \le \cosh^2(\sqrt{\mu_N} t)\varepsilon^2. \tag{24}$$

Consider the first term on the right-hand side of (23) taking it into account that $u \in M$. We have

$$\|u - u^N\|_0^2 = \sum_{k=N+1}^{\infty} |u_k^+|^2 + \sum_{k=N+1}^{\infty} |u_k^-|^2.$$

The relations (11) and (20) yield

$$\|u - u^N\|_0^2 \le \sigma^2(m, N), \tag{25}$$

where

$$\sigma^2(m,N) = \left(\sum_{k=N+1}^{\infty} (1 + 0,5\mu_k)\{\varphi_{0_k}^+\}^2 \right)^{\frac{T-t}{T}} \left(m^2 + \sum_{k=N+1}^{\infty} 0,5\mu_k \{\varphi_{0_k}^+\}^2 \right)^{\frac{t}{T}} e^{2t(T-t)}$$
$$+ \left(\sum_{k=N+1}^{\infty} \{\varphi_{0_k}^-\}^2 \right)^{1-\frac{t}{T}} \left(m^2 + 4T \int_0^T \sum_{k=N+1}^{\infty} \gamma_{3_k}(t)\, dt \right)^{\frac{t}{T}} + 4T \int_0^T \sum_{k=N+1}^{\infty} \gamma_{3_k}(t)\, dt$$

and $\sigma(m, N) \to 0$ as $N \to \infty$. In view of (24) and (25), from (23) we obtain that

$$\|u - u^{N_\varepsilon}\|_0 \le \cosh(\sqrt{\mu_N}t)\varepsilon + \sigma(m, N).$$

As $\varepsilon \to 0$, there exists a choice of the parameter N such that $\cosh(\sqrt{\mu_N}t)\varepsilon + \sigma(m, N)$ tends to zero. Indeed, if

$$\omega(t, \varepsilon) = \inf_N\{\cosh(\sqrt{\mu_N}t)\varepsilon + \sigma(m, N)\}$$

then we can show that

$$\lim_{\varepsilon\to 0} \omega(t, \varepsilon) = 0. \tag{26}$$

Assume that δ is sufficiently small. From the equality $\lim_{N\to\infty} \sigma(m, N) = 0$ it follows that there exists $N(\delta)$ such that $\sigma(m, N) \le \frac{\delta}{2}$ for all $N \ge N(\delta)$. Put $\eta(\delta) = \inf_{N\ge N(\delta)} \cosh(\sqrt{\mu_N}t)$. If $\varepsilon \le \frac{1}{2}\frac{\delta}{\eta(\delta)}$, then $\omega(t, \varepsilon)$ satisfies the inequality $\omega(t, \varepsilon) \le \delta$, which proves (26).

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K. S. Fayazov; I. O. Khajiev
National University of Uzbekistan, Tashkent, Uzbekistan
`kfayazov@yahoo.com`, `h.ikrom@mail.ru`

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ON THE CARDINALITY OF A FINALLY COMPACT T_1–SPACE OF COUNTABLE PSEUDOCHARACTER

P. V. Chernikov

Abstract. A. V. Arkhangel′skiĭ posed the following problem: Estimate the cardinality of a finally compact T_1-space X of countable pseudocharacter. We obtain such an estimate; namely, we prove that $|X| < \beta$, where β is the first measurable cardinal. The estimate is sharp.

Keywords: countably complete ultrafilter, finally compact space

1. Preliminaries. A topological space X is called *finally compact* if each finite covering of X has a countable subcovering of this space.

A topological space X is said to *have countable pseudocharacter* if each point in X is representable as the intersection of countably many open sets.

A ultrafilter D over a set A is called *countably complete*, if $\bigcap_{i=1}^{\infty} A_i$ belongs to D for all $A_i \in D$, $i = 1, 2, \dots$.

The cardinality of a set X is called a *measurable cardinal* if there exists a nontrivial countably complete ultrafilter over X.

The following **problem** is formulated in [1, p. 34]: *Estimate the cardinality of a finally compact T_1-space of countable pseudocharacter.*

At the beginning of the 1980s, I. Juhasz and P. V. Chernikov proved independently that if X is a finally compact T_1-space of countable pseudocharacter then $|X| < \beta$, where β is the first measurable cardinal [2, p. 31; 3]. Juhasz stated this without proof. The author's proof is contained in the rare article [3]. We will provide this assertion with a proof, so solving the above-formulated problem.

Below we will need

Lemma. *Suppose that A is an arbitrary nonempty set, and D is a countably complete ultrafilter over A, while X is a finally compact T_1-space of countable pseudocharacter, $f \in X^A$. Then there exists a unique point $x_0 \in X$ such that, for every neighborhood V of x_0,*

$$\{i : f(i) \in V\} \in D.$$

Proof. Suppose that, for every $x \in X$, there is a neighborhood V_x such that $\{i : f(i) \in V_x\} \notin D$. The space X is finally compact, and so we can refine a countable subcovering $\{V_{x_n}\}_{n=1}^{\infty}$ of the covering $\{V_x\}_{x \in X}$ of X. Since

$$\bigcup_{n=1}^{\infty} \{i : f(i) \in V_{x_n}\} = A,$$

there is a number m for which $\{i : f(i) \in V_{x_m}\} \in D$; a contradiction. Thus, the existence of x_0 is established.

Prove uniqueness. Suppose that there exist two different points $x_1, x_2 \in X$ with the above property. We have

$$\{x_j\} = \bigcap_{n=1}^{\infty} \sigma_n^j, \quad j = 1, 2,$$

where σ_n^j are open subsets in X $(n = 1, 2, \dots)$. Consequently,

$$A_j = \{i : f(i) = x_j\} = \bigcap_{n=1}^{\infty} \{i : f(i) \in \sigma_n^j\} \in D, \quad j = 1, 2.$$

Take $i_0 \in A_1 \cap A_2$. Then $f(i_0) = x_1$, $f(i_0) = x_2$; a contradiction. The lemma is proved.

The point x_0 is called the *D-limit* of $f \in X^A$ [4]. Following [4], we denote this point by D-$\lim f$.

Theorem 1. *Let X be a finally compact T_1-space of countable pseudocharacter. Then $|X|$ is a nonmeasurable cardinal.*

PROOF. Let D be a countably complete ultrafilter over X and let $\mathrm{id} : X \to X$ be the identity mapping. Put $x_0 = D$-$\lim \mathrm{id}$. By hypothesis, there exist open sets σ_n, $n \geq 1$, in X such that $\{x_0\} = \bigcap_{n=1}^{\infty} \sigma_n$. For all $n \geq 1$, we have $\sigma_n \in D$; therefore, $\{x_0\} \in D$; i.e., D is a principal ultrafilter. The theorem is proved.

Thus, if X is a finally compact T_1-space of countable pseudocharacter then $|X| < \beta$, where β is the first measurable cardinal.

This result was known earlier in the case when X is a regular finally compact space of countable pseudocharacter [1, p. 34; 5, Chapter IV, Problem 119].

REMARK. Juhasz proved in [2] that, for every set X_0, $|X_0| < \beta$, there exists a finally compact T_1-space X^* of countable pseudocharacter such that $|X_0| < |X^*| < \beta$. This implies that the (above) estimate $|X| < \beta$ is sharp.

2. Focus on the convergence of D-limits.

Theorem 2. *Suppose that A is an arbitrary nonempty set, and D is a countably complete ultrafilter over A, while X is a finally compact T_1-space of countable pseudocharacter, $\{f_n\}_{n=1}^{\infty} \subset X^A$, $f \in X^A$. For the sequence $\{D\text{-}\lim f_n\}_{n=1}^{\infty}$ to converge to a point D-$\lim f$, it is necessary and sufficient that*

$$S = \{i \in A : \lim_{n\to\infty} f_n(i) = f(i)\} \in D.$$

PROOF. *Necessity:* Suppose that the sequence $\{D\text{-}\lim f_n\}_{n=1}^{\infty}$ converges to D-$\lim f$. Put $a_n = D$-$\lim f_n$, $n \geq 1$, $a_0 = D$-$\lim f$. There exists a countable family of open sets $\{\sigma_k^n\}_{k=1}^{\infty}$ in X such that

$$\{a_n\} = \bigcap_{k=1}^{\infty} \sigma_k^n \quad (n = 0, 1, \dots).$$

We have

$$M_n = \{i \in A : f_n(i) = a_n\} = \bigcap_{k=1}^{\infty} \{i \in A : f_n(i) \in \sigma_k^n\} \in D, \quad n \geq 1,$$

$$M_0 = \{i \in A : f(i) = a_0\} = \bigcap_{k=1}^{\infty} \{i \in A : f(i) \in \sigma_k^0\} \in D.$$

Let $M = \bigcap_{n=0}^{\infty} M_n \in D$. If $i_0 \in M$ then

$$f_n(i_0) = a_n \to a_0 = f(i_0).$$

Therefore, $S \supset M$, and hence $S \in D$.

Sufficiency: Suppose that S belongs to D. Let us show that then the sequence $\{D\text{-}\lim f_n\}_{n=1}^{\infty}$ converges to $D\text{-}\lim f$. Put $a_n = D\text{-}\lim f_n$, $n \geq 1$, $a_0 = D\text{-}\lim f$. There exists a countable family of open sets $\{V_k^n\}_{k=1}^{\infty}$ in X such that

$$\{a_n\} = \bigcap_{k=1}^{\infty} V_k^n \quad (n = 0, 1, \dots).$$

We infer

$$M_n = \{i \in A : f_n(i) = a_n\} = \bigcap_{k=1}^{\infty} \{i \in A : f_n(i) \in V_k^n\} \in D, \quad n \geq 1,$$

$$M_0 = \{i \in A : f(i) = a_0\} = \bigcap_{k=1}^{\infty} \{i \in A : f(i) \in V_k^0\} \in D.$$

We also have

$$M = \bigcap_{n=0}^{\infty} M_n \in D, \quad S \cap M \in D.$$

If $i_0 \in S \cap M$ then $\{D\text{-}\lim f_n\}_{n=1}^{\infty}$ converges to $D\text{-}\lim f$. The theorem is proved.

The convergence of D-limits is also considered in [6, 7].

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P. V. Chernikov
Novosibirsk State University, Novosibirsk

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THE METHOD OF LAGRANGE MULTIPLIERS FOR SOLVING A MODEL CRACK PROBLEM

A. V. Zhil′tsov and R. V. Namm

Abstract. We consider a method for solving a model crack problem constructed from modified Lagrangian functionals. The weak lower semicontinuity of the sensitivity functional is proved. Basing on the proven property, a dual method for solving the model problem is constructed. The results of numerical computations are given.

Keywords: model crack problem, Lagrangian functional, duality method, sensitivity functional, saddle point, convex programming, minimization, numerical experiment

Introduction

The classical approach to describing the problem of the equilibrium of an elastic body with a crack consists in that boundary conditions in the form of equalities on the faces of the crack are given. Many works are devoted to the study of such boundary value problems. At the same time, it is well known that, from the standpoint of applications, the obtained linear models have an obvious defect: the opposite faces of the crack can penetrate each other.

The monograph [1] deals with a more complicated model, in which the faces of the crack cannot penetrate each other. The mutual nonpenetration of the faces of the crack is attained by defining nonlinear boundary conditions on the faces.

Analysis of similar problems can be found in [2–6]. Various approaches and tricks are applied for their solving there; numerical computations are also given.

In this article, we consider the possibility of applying modified Lagrangian functionals to solving a crack problem with mutual nonpenetration. In the general case, solutions to such problems are just H^1-smooth, which does not make it possible to prove the existence of a saddle point for the Lagrangian. However, we can prove that if the dual problem is solvable then an Uzawa-type algorithm converges to the solution in the functional. For the classical Lagrangian functionals, the solvability of the dual problem does not guarantee that.

1. The Modified Duality Scheme

Consider the problem of equilibrium of a membrane containing a cut on whose faces nonlinear boundary conditions are given [1, p. 58]. We assume that $\Omega \subset \mathbb{R}^2$ is a bounded convex domain with boundary Γ, while $\gamma \subset \Omega$ is a continuous nonclosed curve without self-intersections (for definiteness, we consider the case when γ is a rectilinear crack parallel to the axis x_2). Put $\Omega_\gamma = \Omega \backslash \overline{\gamma}$.

It is required to find a function u in the domain Ω_γ such that

$$\begin{gathered} -\triangle u = f \quad \text{in } \Omega_\gamma, \\ u = 0 \quad \text{on } \Gamma, \\ [u] \geq 0, \quad [u_{x_2}] = 0, \quad u_{x_2} \leq 0, \quad u_{x_2}[u] = 0 \quad \text{on } \gamma. \end{gathered} \tag{1}$$

Here $f \in L_2(\Omega)$ is a given function; $u_{x_2} = \frac{\partial u}{\partial x_2}$ is the derivative with respect to the normal to the crack; $[u] = u^+ - u^-$ is the jump of the function u on γ (at every point $x \in \gamma$, the function takes two values: u^+ and u^- corresponding to the top and bottom faces of the crack).

Let

$$H^1_\Gamma(\Omega_\gamma) = \{v \in H^1(\Omega_\gamma) : v = 0 \text{ on } \Gamma\}.$$

Problem (1) corresponds to the problem of the minimization of the energy functional

$$\begin{gathered} J(v) = \frac{1}{2}\int\limits_{\Omega_\gamma} |\nabla v|^2\,dx - \int\limits_{\Omega_\gamma} fv\,dx \to \min_{v\in K}, \\ K = \bigl\{v \in H^1_\Gamma(\Omega_\gamma) : [v] \geq 0 \text{ a.e. on } \gamma\bigr\}. \end{gathered} \tag{2}$$

Given $m \in L_2(\gamma)$, introduce the set

$$K_m = \bigl\{v \in H^1_\Gamma(\Omega_\gamma) : -[v] \leq m \text{ i.e. on } \gamma\bigr\}.$$

If m is bounded below then K_m is nonempty. But if $m \in L_2(\gamma)\backslash H^{1/2}(\gamma)$ is not bounded below then K_m can be empty.

Given $m \in L_2(\gamma)$, define the sensitivity functional

$$\chi(m) = \begin{cases} \inf\limits_{v\in K_m} J(v) & \text{if } K_m \neq \varnothing, \\ +\infty & \text{if } K_m = \varnothing. \end{cases}$$

Its effective domain $\operatorname{dom}\chi = \{m \in L_2(\gamma) : \chi(m) < +\infty\}$ is a convex nonclosed set in $L_2(\gamma)$; moreover, $\overline{\operatorname{dom}\chi} = L_2(\gamma)$.

Under the condition that $m \in \operatorname{dom}\chi$, by the coerciveness of l $J(v)$, the problem

$$\begin{gathered} J(v) = \frac{1}{2}\int\limits_{\Omega_\gamma} |\nabla v|^2\,dx - \int\limits_{\Omega_\gamma} fv\,dx \to \min_{v\in K_m}, \\ K_m = \bigl\{v \in H^1_\Gamma(\Omega_\gamma) : -[v] \leq m \text{ a.e. on } \gamma\bigr\} \end{gathered} \tag{3}$$

has a unique solution which we will denote by $u_m = \operatorname{argmin}_{-[v]\leq m} J(v)$. Then, by definition, $\chi(m) = J(u_m)$, and $\chi(0) = \inf_{-[v]\leq 0} J(v) = J(u)$.

Show that $\chi(m)$ is a convex functional on $\operatorname{dom}\chi$. Suppose that $m', m'' \in L_2(\gamma)$ and $\chi(m') = J(v')$, $\chi(m'') = J(v'')$. We have

$$-[v'] \leq m', \quad -[v''] \leq m''.$$

Multiplying the above inequalities by $(1-\lambda)$ and λ (for $0 \leq \lambda \leq 1$), and summing up the results, we infer

$$-(1-\lambda)[v'] - \lambda[v''] \leq (1-\lambda)m' + \lambda m'', \quad \lambda \in (0,1).$$

Then

$$\begin{gathered} \chi((1-\lambda)m' + \lambda m'') = \inf_{-[v]\leq(1-\lambda)m'+\lambda m''} J(v) \\ \leq J((1-\lambda)[v'] + \lambda[v'']) \leq (1-\lambda)J(v') + \lambda J(v'') \\ = (1-\lambda)\chi(m') + \lambda\chi(m''). \end{gathered}$$

Lemma 1. *If $\{u_i\}$ is a bounded sequence in $H^1(\Omega_\gamma)$ then $\{[u_i]\}$ is a compact sequence in $L_2(\gamma)$.*

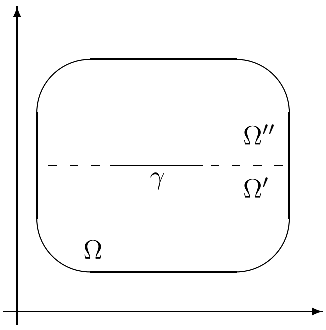


Fig. 1

PROOF. Assume that γ can be extended to the intersection with the exterior boundary Γ so that Ω is partitioned into two subdomains Ω' and Ω'', with Lipschitz boundaries $\partial\Omega'$ and $\partial\Omega''$ respectively and, moreover, $\gamma^+ = \partial\Omega'' \cap \gamma$ and $\gamma^- = \partial\Omega' \cap \gamma$ (Fig. 1). The embeddings $H^1(\Omega') \subset H^{1/2}(\partial\Omega') \subset H^{1/2}(\gamma^-)$ and $H^1(\Omega'') \subset H^{1/2}(\partial\Omega'') \subset H^{1/2}(\gamma^+)$ are continuous; hence, the following estimates for the norms hold:

$$\|u'\|_{H^{1/2}(\gamma^-)} \le C_1 \|u'\|_{H^1(\Omega')},$$
$$\|u''\|_{H^{1/2}(\gamma^+)} \le C_2 \|u''\|_{H^1(\Omega'')},$$

where u' and u'' are the restrictions of some function u (possibly taking different values on γ^- and γ^+) on Ω' and Ω'' respectively.

Summing them up the squares of these inequalities, we get

$$\|u''\|^2_{H^{1/2}(\gamma^+)} + \|u'\|^2_{H^{1/2}(\gamma^-)} \le C_2 \|u''\|^2_{H^1(\Omega'')} + C_1 \|u'\|^2_{H^1(\Omega')}, \quad C_1, C_2 > 0,$$

or else $\|u^+\|^2_{H^{1/2}(\gamma)} + \|u^-\|^2_{H^{1/2}(\gamma)} \le \max\{C_1, C_2\} \|u\|^2_{H^1(\Omega_\gamma)}$.

From the well-known inequality $\|u - v\|^2_X \le 2\big(\|u\|^2_X + \|v\|^2_X\big)$ we infer

$$\|u^+\|^2_{H^{1/2}(\gamma)} + \|u^-\|^2_{H^{1/2}(\gamma)} \ge \frac{1}{2} \|u^+ - u^-\|^2_{H^{1/2}(\gamma)}.$$

Here, on the right-hand side, we have obtained the norm of the jump of the function. Thus, $\|[u]\|_{H^{1/2}(\gamma)} \le C \|u\|_{H^1(\Omega_\gamma)}$, where $C = \sqrt{2\max\{C_1, C_2\}}$.

This means that if $\{u_i\}$ is a bounded sequence in $H^1(\Omega_\gamma)$ then $\{[u_i]\}$ is a bounded sequence in $H^{1/2}(\gamma)$. The space $H^{1/2}(\gamma)$ is compactly embedded in $L_2(\gamma)$, which implies that $\{[u_i]\}$ is a compact sequence in $L_2(\gamma)$. $\square$

It is necessary to clarify that $[u] \in H^{1/2}_{00}(\gamma)$. The norm in $H^{1/2}_{00}(\gamma)$ is defined as follows:

$$\|v\|^2_{H^{1/2}_{00}(\gamma)} = \|v\|^2_{H^{1/2}(\gamma)} + \|\frac{v}{\sqrt{\rho}}\|^2_{L_2(\gamma)},$$

where $\rho(x) = \operatorname{dist}(x, \partial\gamma)$ (see [1, p. 53]).

Theorem 1. *The sensitivity functional $\chi(m)$ is weakly semicontinuous on $L_2(\gamma)$.*

PROOF. Since $\chi(m)$ is convex, for proving the theorem, it suffices to show that it is lower semicontinuous (in norm-convergence) in $L_2(\gamma)$. Take an arbitrary convergent sequence $\{m_i\} \subset L_2(\gamma)$; let $\overline{m} = \lim_{i\to\infty} m_i$. The sensitivity functional $\chi(m)$ is lower semicontinuous if the following are fulfilled:

(1) $\lim\limits_{i\to\infty} \chi(m_i) = +\infty$ for $\overline{m} \notin \operatorname{dom}\chi$;

(2) $\varliminf\limits_{i\to\infty} \chi(m_i) \ge \chi(\overline{m})$ for $\overline{m} \in \operatorname{dom}\chi$.

Consider the two cases consecutively. In proving the theorem, we may confine exposition to a sequence $\{m_i\}$ from the effective domain $\chi(m)$, $m_i \in \operatorname{dom}\chi$, since, outside the domain, the functional takes the value $+\infty$ and the inequality of lower semicontinuity holds.

1. Let $\overline{m} \notin \operatorname{dom}\chi$. Consider the sequence $\{u_{m_i}\}$, where $u_{m_i} = \operatorname{argmin}_{v\in K_{m_i}} J(v)$.

Prove that $\lim_{i\to\infty} \|u_{m_i}\|_{H^1(\Omega_\gamma)} = +\infty$. Suppose the contrary, i.e., suppose that a sequence $\{u_{m_i}\}$ has a bounded subsequence. Assume without loss of generality that $\{u_{m_i}\}$ is itself bounded in $H^1(\Omega_\gamma)$. By Lemma 1, $\{[u_{m_i}]\}$ is a compact sequence in $L_2(\gamma)$. Let $t \in H_{00}^{1/2}(\gamma)$ be a weak limit point of this sequence which, without loss of generality, we will assume to be the weak limit. Then $\{[u_{m_i}]\}$ converges to t in the norm in $L_2(\gamma)$.

Since $m_i \longrightarrow \overline{m}$ in $L_2(\gamma)$ and $u_{m_i} \longrightarrow t$ in $L_2(\gamma)$, the condition $-[u_{m_i}] \leq m_i$ implies that $-t \leq \overline{m}$, and this means that $K_{\overline{m}} \neq \varnothing$ or $\overline{m} \in \operatorname{dom}\chi$. The obtained contradiction shows that $\lim_{i\to\infty} \|u_{m_i}\|_{H^1(\Omega_\gamma)} = +\infty$.

The coerciveness of $J(v)$ yields

$$\lim_{i\to\infty} \chi(m_i) = \lim_{i\to\infty} J(u_{m_i}) = +\infty.$$

2. Suppose now that $\overline{m} \in \operatorname{dom}\chi$. From the sequence $\{m_i\}$ extract a subsequence $\{m_j\} \subset \{m_i\}$ for which

$$\lim_{j\to\infty} \chi(m_j) = \varliminf_{i\to\infty} \chi(m_i).$$

As above, consider the sequence $\{u_{m_j}\}$, where $u_{m_j} = \operatorname{argmin}_{v\in K_{m_j}} J(v)$.

If the sequence $\{u_{m_j}\}$ is not bounded in $H^1(\Omega_\gamma)$ then, by the coerciveness of the functional, $J(u_{m_i}) \longrightarrow +\infty$, and then $\lim_{i\to\infty}\chi(m_i) = +\infty$, and the desired inequality of lower semicontinuity holds.

In the case when the sequence $\{u_{m_j}\}$ is bounded in $H^1(\Omega_\gamma)$, again argue as in the first part of the proof and obtain $-t \leq \overline{m}$.

Let $\tilde{u} = \operatorname{argmin}_{[v]=t \text{ on } \gamma} J(v)$. We have

$$\begin{aligned}
J(u_{m_j}) - J(\tilde{u}) &= \frac{1}{2}\int\limits_{\Omega_\gamma} |\nabla u_{m_j}|^2\, d\Omega - \int\limits_{\Omega_\gamma} f u_{m_j}\, d\Omega - \frac{1}{2}\int\limits_{\Omega_\gamma} |\nabla \tilde{u}|^2\, d\Omega + \int\limits_{\Omega_\gamma} f\tilde{u}\, d\Omega \\
&= \frac{1}{2}\int\limits_{\Omega_\gamma} |\nabla(\tilde{u} + (u_{m_j} - \tilde{u}))|^2\, d\Omega - \frac{1}{2}\int\limits_{\Omega_\gamma} |\nabla\tilde{u}|^2\, d\Omega - \int\limits_{\Omega_\gamma} f(u_{m_j} - \tilde{u})\, d\Omega \\
&= \int\limits_{\Omega_\gamma} \nabla\tilde{u}\nabla(u_{m_j} - \tilde{u})\, d\Omega + \frac{1}{2}\int\limits_{\Omega_\gamma} |\nabla(u_{m_j} - \tilde{u})|^2\, d\Omega - \int\limits_{\Omega_\gamma} f(u_{m_j} - \tilde{u})\, d\Omega \\
&= \langle\Theta, [u_{m_j} - \tilde{u}]\rangle + \frac{1}{2}\int\limits_{\Omega_\gamma} |\nabla(u_{m_j} - \tilde{u})|^2\, d\Omega,
\end{aligned}$$

where

$$\langle\Theta, [v]\rangle = \int\limits_{\Omega_\gamma} \nabla\tilde{u}\nabla v\, d\Omega - \int\limits_{\Omega_\gamma} f v d\Omega;$$

moreover, $\Theta \in H_{00}^{-1/2}(\gamma)$ [1, 7].

Since $\{[u_{m_j}]\}$ converges weakly to t in $H_{00}^{1/2}(\gamma)$, by the uniqueness of the weak limit, we infer

$$\lim_{j\to\infty} \langle\Theta, [u_{m_j} - \tilde{u}]\rangle = 0.$$

Therefore, we have the estimate

$$\lim_{j\to\infty} J(u_{m_j}) \geq J(\tilde{u}) \geq \chi(\overline{m});$$

consequently,

$$\varliminf_{j\to\infty} \chi(m_j) \geq \chi(\overline{m}). \quad \square$$

In $H^1(\Omega_\gamma) \times L_2(\gamma)$, define the modified Lagrangian functional

$$M(v,l) = J(v) + \frac{1}{2r} \int\limits_\gamma (((l - r[v])^+)^2 - l^2)\, d\sigma,$$

where $r = \text{const} > 0$, $(l - r[v])^+ = \max\{0, l - r[v]\}$.

Definition 1. A pair $(v^*, l^*) \in H^1(\Omega_\gamma) \times L_2(\gamma)$ is called a *saddle point* of the functional $M(v,l)$ if the two-sided inequality

$$M(v^*, l) \le M(v^*, l^*) \le (v, l^*), \quad (v,l) \in H^1(\Omega_\gamma) \times L_2(\gamma)$$

holds.

The dual functional for $M(v,l)$ has the two equivalent representations [8]:

$$\underline{M}(l) = \inf_{v \in H^1(\Omega_\gamma)} \left\{ J(v) + \frac{1}{2r} \int\limits_\gamma (((l - r[v])^+)^2 - l^2)\, d\sigma \right\}, \tag{4}$$

$$\underline{M}(l) = \inf_{m \in L_2(\gamma)} \left\{ \chi(m) + \int\limits_\gamma lm\, d\sigma + \frac{r}{2} \int\limits_\gamma m^2\, d\sigma \right\}, \tag{5}$$

where $\chi(m)$ is the above-defined sensitivity functional.

Using the same scheme as in [9], we can prove the following

Theorem 2. *The dual functional $\underline{M}(l)$ is continuous in $L_2(\gamma)$.*

Theorem 3. *The dual functional $\underline{M}(l)$ is Gâteaux differentiable in $L_2(\gamma)$ and its derivative $\nabla\underline{M}(l)$ is Lipschitz continuous with constant $1/r$; i.e.,*

$$\|\underline{M}(l_1) - \underline{M}(l_2)\|_{L_2(\gamma)} \le \frac{1}{r} \|l_1 - l_2\|_{L_2(\gamma)}, \quad l_1, l_2 \in L_2(\gamma).$$

Consider the dual problem

$$\underline{M} \to \max, \quad l \in L_2(\gamma). \tag{6}$$

For solving problem (6), we can use the gradient maximization method [9–11]

$$l^{k+1} = l^k + \theta_k m(l^k), \quad k = 0,1,2,\dots \quad (l^0 \in L_2(\gamma)), \tag{7}$$

where

$$m(l^k) = \operatorname*{argmin}_{m \in L_2(\gamma)} \left\{ \chi(m) + \int\limits_\gamma l^k m\, d\sigma + \frac{r}{2} \int\limits_\gamma m^2\, d\sigma \right\}, \quad \theta_k \in [\beta, 2r - \beta],\ \beta \in (0, r].$$

Theorem 4. *Algorithm* (7) *satisfies the limit equality* [9]

$$\lim_{k \to \infty} \|m(l^k)\|_{L_2(\gamma)} = 0.$$

Algorithm (7) is rewritten as [8]:

$$u^{k+1} = \operatorname*{argmin}_{v \in H^1(\Omega_\gamma)} \left\{ J(v) + \frac{1}{2r} \int\limits_\gamma (((l^k - r[v])^+)^2 - (l^k)^2)\, d\sigma \right\}$$

$$l^{k+1} = l^k + \theta_k \max\left\{ -u^{k+1}, -\frac{l^k}{r} \right\},\ l^0 \in L_2(\gamma),\ \theta_k \in [\beta, 2r - \beta],\ \beta \in (0, r]. \tag{8}$$

Under the condition of the solvability of (6), algorithm (8) converges in the functional; i.e.,

$$\lim_{k \to \infty} J(u^k) = \min_{v \in K} J(v) = J(u^*).$$

Here u^* is the solution to (2).

Indeed, $\chi(m)$ is a weakly lower semicontinuous functional on $L_2(\gamma)$. Therefore,

$$\varliminf_{k\to\infty} \left\{ \chi(m(l^k)) + \int\limits_\gamma l^k m(l^k)\, d\sigma + \frac{r}{2} \int\limits_\gamma m^2(l^k)\, d\sigma \right\}$$
$$= \varliminf_{k\to\infty} \chi(m(l^k)) \geq \chi(0) = J(u^*).$$

On the other hand,

$$\underline{M}(l^k) = \chi(m(l^k)) + \int\limits_\gamma l^k m(l^k)\, d\sigma + \frac{r}{2} \int\limits_\gamma m^2(l^k)\, d\sigma$$
$$= \inf_{m\in L_2(\gamma)} \left\{ \chi(m) + \int\limits_\gamma l^k m\, d\sigma + \frac{r}{2} \int\limits_\gamma m^2\, d\sigma \right\} \leq \chi(0), \quad k = 0, 1, 2, \dots .$$

Therefore,

$$\varlimsup_{k\to\infty} \left\{ \chi(m(l^k)) + \int\limits_\gamma l^k m(l^k)\, d\sigma + \frac{r}{2} \int\limits_\gamma m^2(l^k)\, d\sigma \right\} \leq \chi(0).$$

Consequently, there exists a limit

$$\lim_{k\to\infty} \left\{ \chi(m(l^k)) + \int\limits_\gamma l^k m(l^k)\, d\sigma + \frac{r}{2} \int\limits_\gamma m^2(l^k)\, d\sigma \right\} = \chi(0) = J(u^*).$$

Then Theorem 4 implies that

$$\lim_{k\to\infty} J(u^k) = \lim_{k\to\infty} \chi(m(l^k)) = \chi(0) = J(u^*).$$

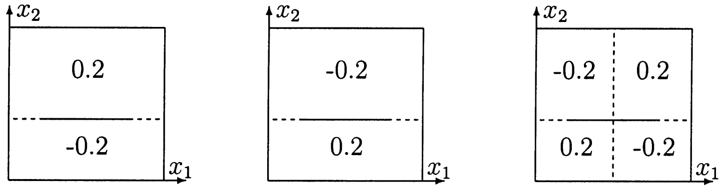
Under the assumption that the solution u^* to the initial problem belongs to $H^2(\Omega_\gamma)$, we can prove [9] that method (8) converges to a saddle point $(u^*, l^*) \in H^1(\Omega_\gamma) \times L_2(\gamma)$ of the Lagrangian functional.

2. A Numerical Experiment by the Finite Element Method

Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1,\ 0 < x_2 < 1\}$ and $\gamma = \{(x_1, x_2) \in \mathbb{R}^2 : 0.2 < x_1 < 0.8,\ x_2 = 0.4\}$. The domain Ω was triangulated with the use of the uniform mesh with meshsize $h = 1/20$. The criteria of finishing the calculations on the internal and external iterations are as follows:

$$\max_i \left| u_i^{(n+1)} - u_i^{(n)} \right| \leq \varepsilon, \quad \max_i \left| l_i^{(n+1)} - l_i^{(n)} \right| \leq 10^2 \varepsilon,$$

respectively, where $\varepsilon = 10^{-8}$. The parameter $r \in \{1, 10, 10^2, 10^3, 10^4\}$. The starting point $(u^{(0)}, l^{(0)})$ is taken to be equal to $(0, 0)$.



(a) f_1 (b) f_2 (c) f_3

Fig. 2

As f, we took piecewise-constant functions. We considered the three different variants of f (Fig. 2) giving fundamentally different solutions.

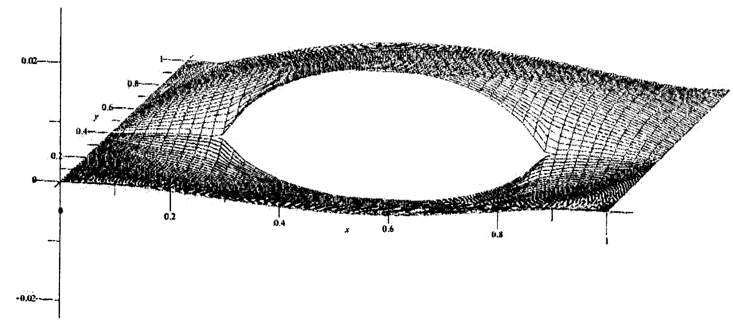


Fig. 3. The form of the plate for f_1

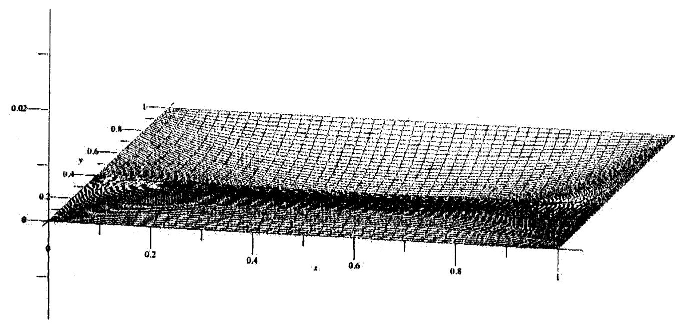


Fig. 4. The form of the plate for f_2

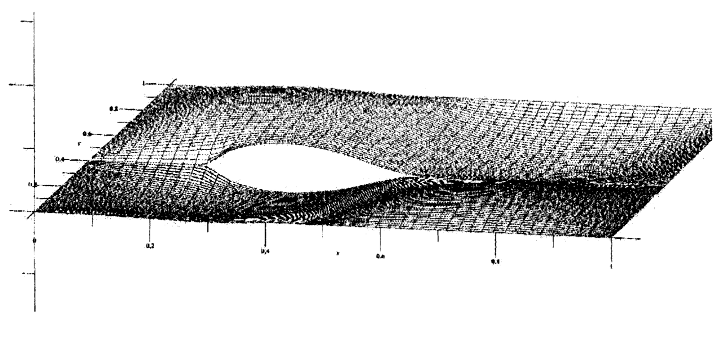


Fig. 5. The form of the plate for f_3

We considered the effect caused by each of the versions of the definition of f, gave the number of external and internal iterations and the implementation time for the demonstration of the complexity of the problems being solved with respect to each other (Table 1).

In the first variant, the faces of the crack diverge completely (Fig. 3), and this means that the dual variable corresponding to the value of the jump of the normal

Table 1. The relative complexity of the problems solved

r	f	Number of internal iterations	Number of external iterations	Implementation time (ms)
1	f_1	438	1	11
	f_2	5172	125	134
	f_3	4510	124	117
10	f_1	348	1	11
	f_2	1277	22	33
	f_3	1108	22	28
10^2	f_1	438	1	11
	f_2	644	6	17
	f_3	536	6	14
10^3	f_1	438	1	11
	f_2	484	4	12
	f_3	404	4	10
10^4	f_1	438	1	11
	f_2	394	3	11
	f_3	342	3	9

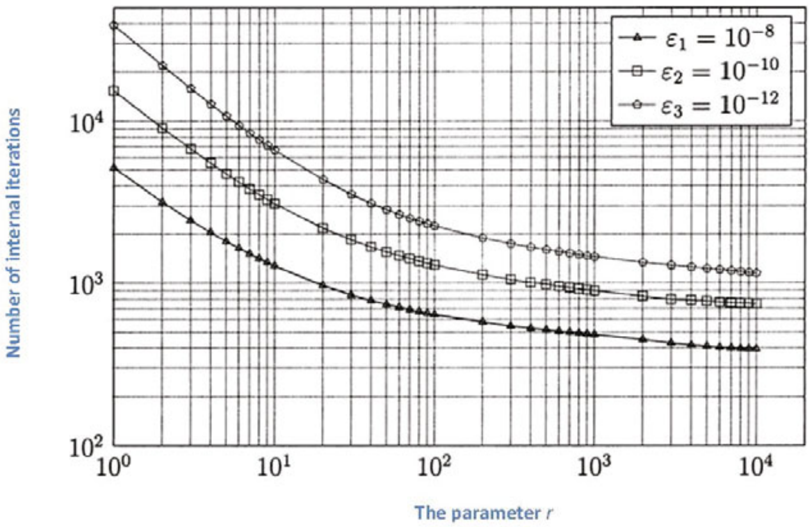


Fig. 6. Dependence of the number of internal iterations on r

derivative on γ, vanishes at all points of γ. Since we initially took the zero function for the dual variable l, only one iteration of the external cycle is implemented.

For the second example, the constraints of the problem do not allow the faces to diverge (Fig. 4), so that the jump of u on γ is zero; in addition, the dual variable takes nonzero values.

In the third variant, the faces of the crack diverged to parts of the crack (Fig. 5).

In studying the modified duality methods, the parameter r can be defined arbi-

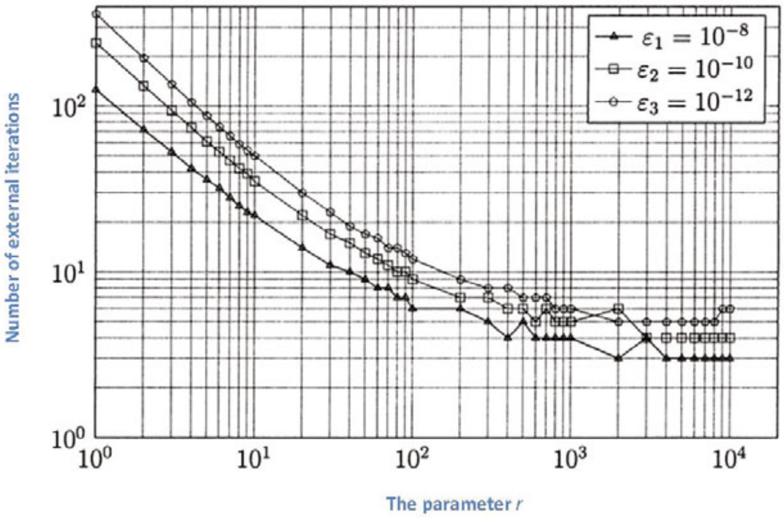


Fig. 7. Dependence of the number of external iterations on r

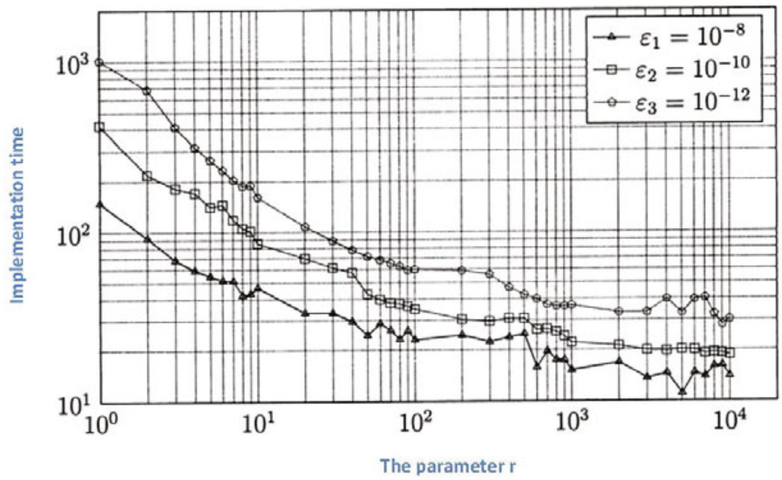


Fig. 8. Dependence of the implementation time on r

trarily; moreover, by Theorem 3, as r increases, the convergence rate of the solution to the dual problem also increases. Figs. 6–8 contain the graphs of the number of internal and external iterations and of the run time of the algorithm depending on r. The graphs are presented in the logarithmic scale. The value of f depicted in Fig. 4 was used. The solution was searched for with accuracies $\varepsilon_1 = 10^{-8}$, $\varepsilon_2 = 10^{-10}$, and $\varepsilon_3 = 10^{-12}$.

The graphs show that the increase of r leads stably to the asymptotic decrease of the number of iterations and the run time of the algorithm.

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A. V. Zhil′tsov
Far Eastern State Transport University, Khabarovsk
`egrevid@gmail.com`

Namm R. V.
Computer Center of the Far Eastern Division of the Russian Academy of Sciences, Khabarovsk
`rnamm@yandex.ru`

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UNIFORM PARTITION OF A SPHERE AND APPLICATION TO COMPUTING THE IRRADIANCE COEFFICIENTS

M. F. Semenov and V. Yu. Shadrin

Abstract. We propose a method of uniform partition of a sphere which can be applied for the numerical integration of surface integrals over the sphere. The results of numerical experiments for calculating the irradiance coefficients are given.

Keywords: sphere, partition, nodes of a cubature formula, irradiance coefficient

Consider the sphere with center the origin $O(0,0,0)$ and radius R. Choose a natural number n and put $\alpha = \alpha_n = \frac{\pi}{2n+1}$. Around the north pole with coordinates $(0,0,R)$, circumscribe a segment with central angle α_n and then partition the upper hemisphere from top downward into n layers with identical central angles α_n. The area of the segment at the pole is equal to

$$S = 2\pi Rh, \quad \text{where } h = R - R\cos\frac{\alpha_n}{2}.$$

If we number the layers from top downward by $i = 1, 2, \dots, n$ then the area of the ith layer equals

$$S_i = 2\pi Rh_i,$$

where $h_i = R\sin(n-i+1)\alpha_n - R\sin(n-i)\alpha_n = 2R\sin\frac{\alpha_n}{2}\sin i\alpha_n$.

Consider the ratio of the area of the ith layer S_i to the area of the pole S:

$$\mu_{ni} = \frac{S_i}{S} = \frac{2\sin i\alpha_n}{\tan\frac{\alpha_n}{4}}.$$

The integer part $[\mu_{ni}]$ of this number means the number of the equiareal sectors with the areas equal to the area of the pole which constitute the ith pole.

Put $\varphi_i = \frac{\pi}{2} - \alpha_n i$ and $\theta_{ij} = \frac{\pi(2j-1)}{\mu_{ni}}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, [\mu_{ni}]$. Define an ordered set of uniformly distributed points on the upper hemisphere $U_N = \{(x_{ij}, y_{ij}, z_{ij})\}$. The coordinates of the points are calculated by the formulas

$$x_{ij} = R\cos\varphi_i\cos\theta_{ij}, \quad y_{ij} = R\cos\varphi_i\sin\theta_{ij}, \quad z_{ij} = R\sin\varphi_i,$$

$$i = 1, 2, \dots, n, \quad j = 1, 2, \dots, [\mu_{ni}].$$

Similarly, define an ordered set of uniformly distributed points on the lower hemisphere $U_S = \{(x_{ij}, y_{ij}, z_{ij})\}$. The coordinates of the points are calculated by the formulas

$$x_{ij} = R\cos\varphi_i\cos\theta_{ij}, \quad y_{ij} = R\cos\varphi_i\sin\theta_{ij}, \quad z_{ij} = -R\sin\varphi_i,$$
$$i = 1, 2, \dots, n, \quad j = 1, 2, \dots, [\mu_{ni}].$$

To these sets, add the north pole P_N with coordinates

$$x_{2n+1,1} = 0, \quad y_{2n+1,1} = 0, \quad z_{2n+1,1} = R$$

and the south pole P_S with coordinates

$$x_{2n+2,1} = 0, \quad y_{2n+2,1} = 0, \quad z_{2n+2,1} = -R.$$

Thus, we obtain an ordered set of points (nodes) that are distributed uniformly over the whole sphere:

$$U = U_N \cup U_S \cup P_N \cup P_S.$$

Theorem 1. *The following hold:*
(1) $\lim\limits_{n\to\infty} \mu_{n1} = 8$;
(2) $\lim\limits_{n\to\infty} \frac{\mu_{nn}}{n} = \frac{16}{\pi}$;
(3) $\frac{16}{\pi} \le \frac{\mu_{ni}}{i} \le 8,\ i = 1, 2, \dots, n.$

Proof. (1) We have

$$\lim_{n\to\infty} \mu_{n1} = \lim_{n\to\infty} \frac{\mu_{n1}}{1} = \lim_{n\to\infty} \frac{2\sin\alpha_n}{\tan\frac{\alpha_n}{4}} = \lim_{n\to\infty} \frac{2\sin\frac{\pi}{2n+1}}{\tan\frac{\pi}{4(2n+1)}} = 8.$$

This means that, in the first layer near the north pole, as $n \to \infty$, the number of the sectors of the area equal to the area of the pole tends to 8. Note that, as was shown in [1], in the planar case, under a similar partition of the disk, the first ring contains exactly 8 sectors of area equal to the area of the central disk.

(2) We infer

$$\lim_{n\to\infty} \frac{\mu_{nn}}{n} = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{2\sin n\alpha_n}{\tan\frac{\alpha_n}{4}} = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{2\sin n\frac{\pi}{2n+1}}{\tan\left(\frac{1}{4}\cdot\frac{\pi}{2n+1}\right)} = 2\cdot \lim_{n\to\infty} \frac{4(2n+1)}{n\pi} = \frac{16}{\pi}.$$

(3) Obviously, $\frac{16}{\pi} \le \frac{\mu_{ni}}{i} \le 8$ for all $i = 1, 2, \dots, n$. The theorem is proved.

Consider an application of the proposed partition for the approximate calculation of the irradiance coefficients in the radiant heat exchange between surfaces one of which is a sphere.

The set of nodes U can be taken as nodes for a cubature formula analogous to the formula proposed in [1, 2].

The irradiance coefficient (angular coefficient) F_{1-2} from surface 1 with area A_1 to surface 2 with area A_2 is defined as follows (see [3]):

$$F_{1-2} = \frac{1}{A_1} \iint\limits_{A_1 A_2} \frac{\cos\beta_1 \cos\beta_2 \, dA_1 dA_2}{\pi R^2}, \tag{1}$$

where R is the distance from the area element dA_1 on A_1 to the area element dA_2 on A_2, β_1, and β_2 are the angles between R and the normal vectors $\vec{N}_1$ and $\vec{N}_2$ to dA_1 and dA_2 respectively directed towards the other surface. The irradiance coefficient shows the share of the radiant flow getting onto surface 2 in the entire flow radiated by surface 1.

Carry out the approximate calculation of the surface integral of the second kind (1) by a cubature formula that is a multidimensional analog of the mean rectangle formula based on the definition of this integral. The idea consists in partitioning the surfaces that are involved in the radiant heat exchange into area elements and choosing the "mean" points of these surfaces as the nodes of the cubature formula. Then the cubature formula takes the form

$$F_{1-2} = \frac{1}{\pi A_1} \sum_{i_1=1}^{n_1} \sum_{j_1=1}^{m_1} \sum_{i_2=1}^{n_2} \sum_{j_2=1}^{m_2} f(M_{i_1 j_1}, M_{i_2 j_2}) \Delta s_{i_1 j_1} \Delta s_{i_2 j_2}, \tag{2}$$

where $n_1 \times m_1$ is the number of area elements on surface 1, $n_2 \times m_2$ is the number of area elements on surface 2, $M_{i_1 j_1} \in \Delta s_{i_1 j_1}$ and $M_{i_2 j_2} \in \Delta s_{i_2 j_2}$ are the nodes of the cubature formula,

$$f(M_{i_1 j_1}, M_{i_2 j_2}) = \frac{\cos\beta_1 \cos\beta_2}{R_{12}^2},$$
$$\vec{R}_{12} = \overrightarrow{M_{i_1 j_1} M_{i_2 j_2}}, \quad \vec{R}_{21} = \overrightarrow{M_{i_2 j_2} M_{i_1 j_1}},$$
$$\beta_1 = \angle(\vec{N}_1, \vec{R}_{12}), \quad \beta_2 = \angle(\vec{N}_2, \vec{R}_{21}),$$

$\Delta s_{i_1 j_1}$ and $\Delta s_{i_2 j_2}$ are the areas of the area elements. The cosines are calculated through the inner product:

$$\cos\beta_1 = \frac{\vec{N}_1 \cdot \vec{R}_{12}}{|\vec{N}_1||\vec{R}_{12}|}, \quad \cos\beta_2 = \frac{\vec{N}_2 \cdot \vec{R}_{21}}{|\vec{N}_2||\vec{R}_{21}|}.$$

The four-fold summation (2) includes only the summands with angles involved in the radiant heat exchange, i.e., only the summands for which $\cos\beta_1 > 0$, $\cos\beta_2 > 0$.

Table 1 contains the results of the computations for embedded concentric spheres with common center and radii $R_1 = 2$ and $R_2 = 1$. Obviously, the irradiance coefficient of the exterior sphere from the interior sphere is equal to 1. The first column contains the values of the area of the area elements ΔS. In the second column, UP stands for the "uniform partition," GP designates the "geographic partition." Under the uniform partition (Fig. 1), all area elements have the almost identical indicated area; under the geographic partition (Fig. 2), the areas situated above and below the "equator" have the almost identical indicated area; the remaining area elements obviously have lesser area. However, a refinement of the mesh near the "poles" in the geographic partition does not lead to an improvement of the computation results. As is seen from Table 1, the cubature formula converges as the number of the nodes of the partition of the sphere grows; here the uniform partition is preferable not only as regards the computation rate but also the exactness of the cubature formula.

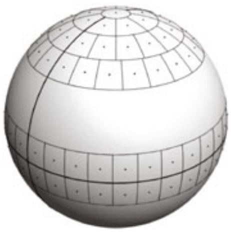

Fig. 1. The uniform partition

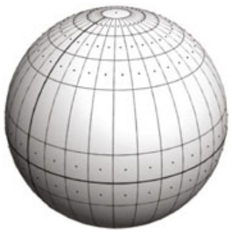

Fig. 2. The geographic partition

Table 2 contains the results of the test problem of calculating the irradiance coefficient from a sphere of radius $R = 1$ onto the interior surface of a cube containing the sphere. The centers of the cube and the sphere coincide, and the edge of the cube is equal to 4.

Table 1. Numerical results for embedded spheres

Area ΔS	Partition method	Number of nodes S_1	Number of nodes S_2	Error	Computation time in seconds
0.0671150	UP	726	202	0.000115448	< 1
	GP	2354	589	0.005913457	< 1
0.0082660	UP	6060	1574	0.000002844	< 1
	GP	19105	4777	0.000800807	4
0.0046685	UP	10676	2746	0.000007244	1
	GP	33826	8457	0.000463989	11
0.0021060	UP	23788	6060	0.000000447	7
	GP	74984	18747	0.000210775	67
0.0011890	UP	42094	10676	0.000000200	20
	GP	132813	33204	0.000119545	130
0.0006285	UP	79290	20022	0.000000145	70
	GP	251256	62815	0.000063867	480
0.0003915	UP	128128	32294	0.000000122	190
	GP	403356	100840	0.000039819	1335
0.0001925	UP	260866	65590	0.000000106	780
	GP	820332	205084	0.000019831	5460

By the closedness property of the irradiance coefficient, the sum of the irradiance coefficients onto separate faces of the cube must equal 1. Each of the faces of the cube was partitioned into 400×400 elementary squares of area 0.0001, an elementary sector of the sphere has area 0.0001181 in the uniform partition, the greatest elementary sector near the "equator" in the geographic partition has the same area 0.0001181. Here F_{S-i} stands for the irradiance coefficient of the sphere onto the ith face, and ε is the computation error.

Table 2. Numerical results for a sphere embedded in a cube

	Uniform partition	Geographic partition
F_{S-1}	0.1667937281	0.1666543690
F_{S-2}	0.1667937281	0.1666543690
F_{S-3}	0.1666035189	0.1666656458
F_{S-4}	0.1666035191	0.1666656458
F_{S-5}	0.1666035189	0.1666656457
F_{S-6}	0.1666035191	0.1666656458
Sum	1.0000015322	0.9999713211
ε	0.0000015322	-0.0000286789

As we see, the results of the numerical experiment also show that the application of the nodes of the above-proposed uniform partition is preferable to the usual geographic partition.

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Semenov M. F.; Shadrin V. Yu.
North-Eastern Federal University, Yakutsk, Russia
sem.mi@mail.ru; vshadr@mail.ru

Address for correspondence:
Ammosov North-Eastern Federal University, Belinskii St., 58, Yakutsk, 677000
Phone: 8(4112)32-14-99, Fax: 8(4112)36-43-47;
http://s-vfu.ru/universitet/rukovodstvo-i-struktura/instituty/niim/mzsvfu/
e-mail: prokopevav85@gmail.com; madu@ysu.ru: ivanegorov51@mail.ru