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ADDRESS FOR CORRESPONDENCE:

Ammosov North-Eastern Federal University, Belinskii St., 58, Yakutsk, 677000
Phone: 8(4112)32-14-99, Fax: 8(4112)36-43-47;
http://s-vfu.ru/universitet/rukovodstvo-i-struktura/instituty/niim/mzsvfu/
e-mail: prokopevav85@gmail.com; madu@ysu.ru: ivanegorov51@mail.ru

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APPLICATION OF THE MODIFIED GALERKIN METHOD TO THE FIRST BOUNDARY VALUE PROBLEMS FOR A MIXED TYPE EQUATION

I. E. Egorov

Abstract. We consider the first boundary problem for a second order mixed type equation of elliptic or hyperbolic type near the bases of a cylindrical domain of the space R^{n+1}. To study the first boundary value problem, we employ the modified Galerkin method and the regularization method. An approximate solution to the first boundary value problem is constructed with the use of a solution to an appropriate boundary value problem for third order system of ordinary differential equations. The error estimate of the modified Galerkin method is established through the regularization parameter and the eigenvalues of the Laplace operator in the space variables with the Dirichlet boundary conditions.

Keywords: Galerkin method, mixed type equation, first boundary value problem, a priori estimate, error estimate, regularization

Many articles are devoted to the study of boundary value problems for mixed type equations (see [1–12]). We can refer to [13] for a brief survey of these and more modern articles in this area. Note that the Galerkin method is universal and widely used for solving boundary value problems for linear and nonlinear equations of mathematical physics [14–16]. The error estimates of the Galerkin method for elliptic and parabolic equations are collected in [15, 16]. On the other hand, the Galerkin method combined with regularization has long been applied to the study of boundary value problems for mixed type equations (see [8, 9, 11]). The stationary Galerkin method is employed in the study of the first boundary value problem for a mixed type equation when the equation is elliptic near the bases of a cylindrical domain in [17]. Error estimates for the stationary Galerkin method through the eigenvalues of the Laplace operator in space and time variables are obtained in [18]. Some particular cases of the Vragov problem [10] are examined in [13, 19]. In this case the modified (nonstationary) Galerkin method [20] is involved together with the regularization method. For these cases the error estimates of the modified Galerkin method are established through the regularization parameter and the eigenvalues of the Laplace operator in space variables with the Dirichlet boundary conditions.

In this article we consider the first boundary value problem for a mixed type equation of the second order (see [12]) when the equation has elliptic or hyperbolic type near the bases of a cylindric domain. To study the first boundary value problem, we employ the modified Galerkin method [13, 19, 20] together with regularization. An approximate solution to the first boundary value problem is constructed by using a solution to a system of ordinary differential equations of the third order.

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Next, we establish an error estimate of the modified Galerkin method through the regularization parameter and eigenvalues of the Laplace operator in spatial variables.

Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $S \in C^2$, $Q = \Omega \times (0,T)$, $S_T = S \times (0,T)$, and $\Omega_t = \Omega \times \{t\}$, $0 \le t \le T$.

Consider the mixed type equation

$$Lu \equiv k(x,t)u_{tt} - \Delta u + a(x,t)u_t + c(x)u = f(x,t), \tag{1}$$

with sufficiently smooth coefficients. Introduce the sets

$$P_0^{\pm} = \{(x,0) : k(x,0) \gtrless 0,\ x \in \Omega\}, \quad P_T^{\pm} = \{(x,T) : k(x,T) \gtrless 0,\ x \in \Omega\}.$$

Boundary Value Problem I. *Find a solution to* (1) *in* Q *satisfying*

$$u|_{S_T} = 0, \quad u|_{t=0} = 0, \quad u_t|_{\overline{P_0^+}} = 0, \quad u|_{\overline{P_T^-}} = 0. \tag{2}$$

The boundary value problem (1), (2) was firstly studied by A. N. Terekhov in [12] by the regularization method.

Let C_L be the space of smooth functions satisfying (2).

Lemma 1 (see [9, 12]). *Assume that* $c(x) > 0$ *is sufficiently large and*

$$k(x,T) < 0, \quad a - \frac{1}{2}k_t \ge \delta > 0.$$

Then there exists a nontrivial infinitely differentiable functions $\varphi(t)$ *and* $\psi(t)$ *such that*

$$(Lu, \varphi u_t + \psi u) \ge C_1 \|u\|_1^2, \quad C_1 > 0,$$

for all $u \in C_L$.

Proof. There exists a positive $T_0 < T$ such that

$$k(x,t) \le -\delta_1, \quad t \in [T_0, T].$$

Choose $\varphi(t), \psi(t) \in C^\infty[0,T]$ such that

$$\varphi(t) = \mu,\ t \in [0,T_0], \quad \varphi'(t) \le 0, \quad \varphi(T) = 0,$$

$$\psi(t) = 1 - \frac{1}{2}\varphi'(t).$$

Let $u(x,t)$ belong to C_L. Integrating by parts, we obtain

$$(Lu, \varphi u_t + \psi u) = \int\limits_Q \left\{ \left[\left(a - \frac{1}{2}k_t\right)\varphi - k\left(\psi + \frac{1}{2}\varphi_t\right)\right] u_t^2 + \left(\psi - \frac{1}{2}\varphi_t\right) \sum_{i=1}^n u_{x_i}^2 \right.$$
$$\left. + c\left(\psi - \frac{1}{2}\varphi_t\right)u^2 + [a\psi - (k\psi)_t]u_t u \right\} dQ + I, \tag{3}$$

where

$$I = -\frac{\mu}{2} \int\limits_{\overline{P_0^-}} k u_t^2 \, dx \ge 0.$$

Now we can choose $\mu > 0$ so that

$$\delta\mu - \max_{\overline{Q}} |k| \ge \delta_2 > 0.$$

In this case
$$\left(a-\frac{1}{2}k_t\right)\varphi-k\left(\psi+\frac{1}{2}\varphi_t\right)\geq\min\{\delta_1,\delta_2\}.$$
Relation (3), the Cauchy inequality, and the conditions of Lemma 1, justify the a priori estimate of Lemma 1.

If $\varepsilon>0$ then $L_\varepsilon u\equiv-\varepsilon D_t^3u+Lu$. We take solutions to the spectral problem
$$-\Delta\varphi=\lambda\varphi,\quad x\in\Omega,\quad \varphi|_S=0$$
as basis functions. The functions $\varphi_k(x)$ form an orthonormal basis for $L_2(\Omega)$ and the corresponding eigenvalues λ_k are such that $0<\lambda_1\leq\lambda_2\leq\cdots$ and $\lambda_k\to+\infty$ as $k\to\infty$ [14].

In what follows, we assume that $k(x,T)<0$. An approximate solution $u^{N,\varepsilon}(x,t)$ to the boundary value problem (1), (2) is sought in the form
$$u^{N,\varepsilon}(x,t)=\sum_{k=1}^{N}c_k^{N,\varepsilon}(t)\varphi_k(x)\equiv v(x,t),$$
where $c_k^{N,\varepsilon}(t)$ are solutions to the following boundary value problem for a system of ordinary differential equations (ODEs) of the third order:
$$(L_\varepsilon u^{N,\varepsilon},\varphi_l)_0=(f,\varphi_l)_0,\tag{4}$$
$$c_l^{N,\varepsilon}(0)=0,\quad D_t^2c_l^{N,\varepsilon}\big|_{t=0}=0,\quad c_l^{N,\varepsilon}(T)=0,\quad l=\overline{1,N},\tag{5}$$
for $k(x,0)<0$ or
$$c_l^{N,\varepsilon}(0)=0,\quad D_tc_l^{N,\varepsilon}\big|_{t=0}=0,\quad c_l^{N,\varepsilon}(T)=0,\quad l=\overline{1,N},\tag{5}$$
for $k(x,0)>0$.

Lemma 2. *Assume that the conditions of Lemma* 1 *and one of the inequalities* $k(x,0)<0$ *or* $k(x,0)>0$ *hold.*

Then there exits a number $\varepsilon_0>0$ *such that approximate solutions to* (1), (2) *satisfy the estimate*
$$\varepsilon\int\limits_Q\varphi v_{tt}^2\,dQ+\|v\|_1^2\leq C_2\|f\|^2,\quad C_2>0,\ 0<\varepsilon\leq\varepsilon_0.\tag{6}$$

PROOF. Without loss of generality, we can assume that $\varphi'(T)=0$. Then from (4) and (5) it follows that
$$(f,\varphi v_t+\psi v)=\varepsilon\int\limits_Q\varphi v_{tt}^2\,dQ-\frac{\varepsilon}{2}\int\limits_Q\left[(\varphi_{tt}+3\psi_t)v_t^2+\psi_{ttt}v^2\right]dQ$$
$$+\frac{1}{2}\varepsilon\int\limits_\Omega\psi v_t^2\,dx\Big|_{t=0}^{t=T}+(Lv,\varphi v_t+\psi v).\tag{7}$$
Note that v satisfy (3).

It suffices to consider only the case of $k(x,0)\leq-\delta_3<0$. Choose $\varepsilon_0>0$ so that $\varepsilon_0\leq\delta_3\mu$. In this case
$$I-\frac{1}{2}\varepsilon\int\limits_{\Omega_0}\psi v_t^2dx\geq0.$$
Decreasing ε_0, if need be, and accounting for (3) and (7); we obtain the a priori estimate (6). Lemma 2 is proven.

Lemma 3. *Assume that $c(x) > 0$ is sufficiently large, the conditions*

$$a - \frac{1}{2}k_t \ge \delta > 0, \quad a + \frac{1}{2}k_t \ge \delta > 0, \quad f, f_t \in L_2(Q),$$

are fulfilled, and one of the following cases $k(x,0) < 0$ and $k(x,T) < 0$, or $k(x,0) > 0$ and $k(x,T) < 0$, $f(x,0) = 0$ holds.

Then there exits a number $\varepsilon_0 > 0$ such that approximate solutions to (1), (2) satisfy the estimate

$$\int\limits_Q \left[v_{tt}^2 + \sum_{i=1}^n v_{tx_i}^2\right] dQ \le C_3[\|f\|^2 + \|f_t\|^2], \quad C_3 > 0,\ 0 < \varepsilon < \varepsilon_0. \tag{8}$$

Proof. For nonnegative infinitely differentiable functions $\xi(t)$ and $\eta(t)$, from (4), (5) it follows that

$$-(f, \xi v_{ttt} + \eta v_{tt}) = \varepsilon \int\limits_Q \xi v_{ttt}^2\, dQ - \frac{\varepsilon}{2}\int\limits_Q \eta_t v_{tt}^2\, dQ$$

$$+\int\limits_Q \left\{\left[\left(a + \frac{1}{2}k_t\right)\xi - k\left(\eta - \frac{1}{2}\xi_t\right)\right]v_{tt}^2 + \left(\eta - \frac{3}{2}\xi_t\right)\sum_{i=1}^n v_{tx_i}^2 + [(a\xi)_t - a\eta + c\xi]v_{tt}v_t\right.$$

$$\left. + c(\xi_t - \eta)v v_{tt} - (\eta_t - \xi_{tt})\sum_{i=1}^n v_{tx_i}v_{x_i}\right\} dQ + J, \tag{9}$$

where

$$J \equiv \int\limits_\Omega \left[\frac{1}{2}(\varepsilon\eta - k\xi)v_{tt}^2 - a\xi v_{tt}v_t + \frac{1}{2}\xi\sum_{i=1}^n v_{tx_i}^2\right] dx\Bigg|_{t=0}^{t=T}.$$

First, we examine the case of $k(x,0) < 0$ and $k(x,T) < 0$. There exist positive numbers t_0, T_0 such that $t_0 < T_0 < T$ and $k(x,t) \le -\delta_1 < 0$, $t \in [0,t_0] \cup [T_0,T]$.

Choose functions $\xi(t)$, $\eta(t)$ such that

$$\xi(0) = \xi(T) = 0,\ \xi'(t) \ge 0,\ t \in [0,t_0],\ \xi(t) = \mu,\ t \in [t_0,T_0],\ \xi'(t) \le 0,\ t \in [T_0,T],$$

$$\eta(t) = 1 + \frac{1}{2}\xi_t,\ t \in [0,t_0], \quad \eta(t) = 1,\ t \in [t_0,T_0], \quad \eta(t) = 1 - \frac{1}{2}\xi_t,\ t \in [T_0,T].$$

Take $\mu > 0$ so that

$$\delta\mu - \max_{\overline{Q}}|k| \ge \delta_2 > 0.$$

In this case

$$\left(a + \frac{1}{2}k_t\right)\xi - k\left(\eta - \frac{1}{2}\xi_t\right) \ge \min\{\delta_1, \delta_2\}. \tag{10}$$

Since

$$J = \frac{\varepsilon}{2}\int\limits_{\Omega_T} \eta v_{tt}^2\, dx$$

is nonnegative; the Cauchy inequality, (6), (10), and (9) imply the a priori estimate (8).

Proceed with the case of $k(x,0) > 0$ and $k(x,T) < 0$. We assume that $k(x,t) \le -\delta_1$, $t \in [T_0,T]$, and $k(x,0) \ge \delta_3 > 0$. Choosing $\mu > 0$, as above, we justify (10).

Using the inequality

$$J \geq \frac{1}{2}(\delta_3\mu - \varepsilon)\int\limits_{\Omega_0} v_{tt}^2\,dx,$$

the Cauchy inequality and (9) and decreasing ε_0, if need be, we can justify (8). Lemma 3 is proven.

Lemma 4. *Let the conditions of Lemma* 3 *hold. Then there exists a number* $\varepsilon_0 > 0$ *such that approximate solutions to* (1), (2) *satisfy the estimate*

$$\|\Delta v\|^2 \leq C_4(\|f\|^2 + \|f_t\|^2), \quad C_4 > 0,\ 0 < \varepsilon \leq \varepsilon_0. \tag{11}$$

Proof. Choose $\varphi(t)$ in $C^\infty[0,T]$ such that $\varphi(0) = \mu > 0$, $\varphi'(t) \leq 0$, and $\varphi(T) = \varphi'(0) = \varphi'(T) = 0$. Relations (4) and (5), the properties of $\varphi_k(x)$, and integration by parts yield

$$-(f, \varphi\Delta v_t + \Delta v) = \varepsilon\int\limits_Q \left[\varphi\sum_{i=1}^n v_{ttx_i}^2 - \frac{1}{2}\varphi_{tt}\sum_{i=1}^n v_{tx_i}^2\right] dQ$$

$$+\int\limits_Q \left\{\left(1 - \frac{1}{2}\varphi_t\right)(\Delta v)^2 + \left[a\varphi - \frac{1}{2}(k\varphi)_t\right]\sum_{i=1}^n v_{tx_i}^2 + \varphi\sum_{i=1}^n [k_{x_i}v_{tt} + a_{x_i}v_t + (cv)_{x_i}]v_{tx_i}\right.$$

$$\left. -(kv_{tt} + av_t + cv)\Delta v\right\} dQ + K, \tag{12}$$

where

$$K = \frac{1}{2}\int\limits_\Omega \left[(k\varphi + \varepsilon)\sum_{i=1}^n v_{tx_i}^2\right] dx\Big|_{t=0}^{t=T}.$$

Consider the case of $k(x,0) \leq -\delta_1 < 0$. Choose $\mu > 0$ so that $\varepsilon_0 \leq \delta_1\mu$. Since K is nonnegative; the Cauchy inequality, the a priori estimates (6), (8), and (12) validate (11).

For $k(x,0) > 0$, we have $v_t(x,0) = 0$, and thus $K \geq 0$ as before. Hence, the estimate (11) is true. Lemma 4 is proven.

Theorem 1. *Assume that* $c(x) > 0$ *is sufficiently large, the conditions*

$$a - \frac{1}{2}|k_t| \geq \delta > 0, \quad f, f_t \in L_2(Q),$$

are fulfilled, and one of the following holds: either $k(x,0) < 0$ *and* $k(x,T) < 0$, *or* $k(x,0) > 0$ *and* $k(x,T) < 0$, $f(x,0) = 0$.

Then (1), (2) *has the unique solution* $u(x,t)$ *in* $W_2^2(Q)$ *satisfying*

$$\|u\|_2 \leq C_5(\|f\| + \|f_t\|), \quad C_5 > 0.$$

Proof. Inequalities (6), (8), (11) and the second basic inequality for the Laplace operator [14, 21] imply that

$$\|u^{N,\varepsilon}\|_2 \leq C_5(\|f\| + \|f_t\|), \quad C_5 > 0 \tag{13}$$

for approximate solutions to (1), (2).

This estimate imply the existence of a solution to (1), (2). Uniqueness of a solution to (1), (2) is guaranteed by Lemma 1. Theorem 1 is proven.

Theorem 2. *Let the conditions of Theorem* 1 *hold. Then the error estimate of the modified Galerkin method is estimated as*

$$\|u - u^{N,\varepsilon}\|_1 \le C_6(\|f\| + \|f_t\|)\big(\varepsilon^{1/2} + \lambda_{N+1}^{-1/4}\big), \quad C_6 > 0, \tag{14}$$

where $u(x,t)$ is an exact solution to (1), (2).

Proof. Consider the functions $\varphi(t)$ and $\psi(t)$ of the proof of Lemma 1. Introduce the manifold of the following subspace of the $L_2(Q)$ space

$$H_N = \left\{\eta(x,t) = \sum_{l=1}^{N} a_l(t)\varphi_l(x) : a_l \in W_2^2(0,T),\ a_l(0) = a_l(T) = 0,\ l = \overline{1,N}\right\}$$

for $k(x,0) < 0$ or

$$H_N = \left\{\eta(x,t) = \sum_{l=1}^{N} a_l(t)\varphi_l(x) : a_l \in W_2^2(0,T),\ a_l(0) = a_l'(0) = a_l(T) = 0,\ l = \overline{1,N}\right\}$$

for $k(x,0) > 0$.

Equations (1) and (4) and the definition of H_N easily imply that

$$(L_\varepsilon u^{N,\varepsilon}, \varphi\eta_t + \psi\eta) = (f, \varphi\eta_t + \psi\eta), \quad (Lu, \varphi\eta_t + \psi\eta) = (f, \varphi\eta_t + \psi\eta), \quad \eta \in H_N,$$

where $u(x,t)$ is an exact solution to (1), (2), ensured by Theorem 1.

Hence,

$$(L(u - u^{N,\varepsilon}), \varphi\eta_t + \psi\eta) = -\varepsilon(u_{ttt}^{N,\varepsilon}, \varphi\eta_t + \psi\eta), \quad \eta \in H_N.$$

The last equality for $\eta = \omega - u^{N,\varepsilon}$ and an arbitrary function ω in H_N takes the form

$$\begin{aligned}
&\big(L(u - u^{N,\varepsilon}), \varphi\big(u_t - u_t^{N,\varepsilon}\big) + \psi(u - u^{N,\varepsilon})\big) \\
&= \varepsilon\big(u_{tt}^{N,\varepsilon}, \big(\varphi\big(\omega_t - u_t^{N,\varepsilon}\big)\big)_t + (\psi(\omega - u^{N,\varepsilon}))_t\big) \\
&\quad + (f - Lu^{N,\varepsilon}, \varphi(u_t - \omega_t) + \psi(u - \omega)).
\end{aligned} \tag{15}$$

Consider the Fourier series

$$u(x,t) = \sum_{k=1}^{\infty} c_k(t)\varphi_k(x), \quad c_k(t) = (u, \varphi_k)_0.$$

If

$$\omega = \sum_{k=1}^{N} c_k(t)\varphi_k(x)$$

then by analogy with [20] we can establish that

$$\|u - \omega\|^2 \le C_7\lambda_{N+1}^{-2}(\|f\|^2 + \|f_t\|^2), \quad C_7 > 0, \tag{16}$$

$$\|u_t - \omega_t\|^2 \le C_8\lambda_{N+1}^{-1}(\|f\|^2 + \|f_t\|^2), \quad C_8 > 0. \tag{17}$$

Lemma 1, along with (13), (16), (17) and (15), validates (14) for the error of the Galerkin method. Theorem 2 is proven.

Remark 1. If $k(x,0) > 0$ and $k(x,T) \ge 0$ or $k(x,0) < 0$ and $k(x,T) \ge 0$, then (1), (2) coincides with the Vragov problem [10] and the results similar to those in [13, 19] are valid.

Remark 2. We can consider a more general elliptic operator of second order instead of the Laplace operator (see [14]).

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I. E. Egorov
North-Eastern Federal University
Institute of Mathematics, Yakutsk, Russia
`IvanEgorov51@mail.ru`

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CONSTRUCTION OF ALMOST PERIODIC SOLUTIONS TO SOME SYSTEMS OF DIFFERENTIAL EQUATIONS

M. F. Kulagina and E. A. Mikishanina

Abstract. We propose a method for constructing Bohr almost periodic solutions to boundary value problems for systems of partial differential equations that arise in solving certain problems for inhomogeneous media.

Keywords: differential equation, boundary value problem, generalized discrete Fourier transform, Fourier series

We consider boundary value problems for systems of differential equations that appear in solving planar problems in the theory of inhomogeneous media in elasticity theory, filtration theory, diffusion theory, heat conduction, electro- and magnetodynamics in the case when the domain is a two-layer (or an l-layer) strip. The boundary conditions are defined both on the boundary of the strip and on the gluing line.

In the general case, the problem looks as follows: find functions $u_{km}(x, y)$ such that, in each of the m strips $-\infty < x < +\infty$, $a_m < y < b_m$, they satisfy the system

$$\sum_{k=1}^{n}\biggl(a_{kj}^{(m)}\frac{\partial^2 u_{km}(x,y)}{\partial x^2}+b_{kj}^{(m)}\frac{\partial^2 u_{km}(x,y)}{\partial y^2}+c_{kj}^{(m)}\frac{\partial u_{km}(x,y)}{\partial x}$$
$$+d_{kj}^{(m)}\frac{\partial u_{km}(x,y)}{\partial y}+e_{kj}^{(m)}u_{km}(x,y)\biggr)=F_{jm}(x,y),\quad j=\overline{1,n},\quad m=\overline{1,l}. \tag{1}$$

The boundary conditions are defined on the boundary of the strip, and gluing conditions are defined on the separation lines. For example, for the two-layer strip, $-\infty < x < +\infty$, $-1 < y < 0$, $0 < y < 1$, these conditions can look as follows: ($k = \overline{1, n}$, $m = \overline{1, 2}$):

$$u_{km}(x,1)=\Phi_{km}(x),$$
$$u_{km}(x,-1)=\Psi_{km}(x),$$
$$u_{k1}(x,0)=u_{k2}(x,0),$$
$$\frac{\partial u_{k1}(x,0)}{\partial y}=\frac{\partial u_{k2}(x,0)}{\partial y},$$

The number of conditions depends on the order of the system (the maximal order of the system is $2n$).

We will search for Bohr almost periodic solutions on every straight line $y =$ const. These solutions will be constructed with the use of the generalized discrete Fourier transform, introduced and studied in [1–3].

Recall the main notions related to almost periodic functions. An *almost periodic* (a.p.) *polynomial* is a function $p(t)$, $-\infty < t < \infty$, that is a linear combination of functions of the form $e^{i\lambda t}$, where $\lambda \in \mathbb{R}$. Denote by Π_C the closure of the set of all a.p. polynomials in the norm of $L_\infty(-\infty, +\infty)$. The set Π_C is the subalgebra in $L_\infty(-\infty, +\infty)$ consisting of all Bohr a.p. functions. Denote by Π_W the set Π_C consisting of the functions $A(t)$ of the form

$$A(t) = \sum_{n=1}^{\infty} a_n e^{i\lambda_n t} \tag{1.1}$$

satisfying the condition $\sum_{n=1}^{\infty} |a_n| < \infty$. The set Π_W is a Banach algebra.

To each $A(t)$ in Π_W, assign the function

$$a(\lambda) = M\left\{A(t)e^{-i\lambda t}\right\} = \lim_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} A(t)e^{-i\lambda t}. \tag{1.2}$$

Such a function exists and can be nonzero for an at most countable set of constraints λ: λ_1, λ_2, K: $a(\lambda_n) = a_n \neq 0$. Thus, to each function in Π_W, assign the function $a(\lambda)$ or the sequence of pairs $a(\lambda) = \{(a_1, \lambda_1), (a_2, \lambda_2), K\}$, where $a_n \in C$ and $\lambda_n \in \mathbb{R}$.

If $A(t) \in \Pi_W$ then the sequence $\{a_n\}$ corresponding to this function belongs to l_1 (we say that $a(\lambda)$ belongs to l_1). Conversely, for every function $a(\lambda) \in l_1$, there exists a function $A(t)$ for which (1.2) holds and $A(t)$ has the form (1.1). Series (1.1) converges absolutely and uniformly for $-\infty < t < \infty$. Consequently, we have established a one-to-one correspondence between functions in Π_W and two-dimensional sequences $a(\lambda) \in l_1$.

Refer to equality (1.1) by which to a sequence $a(\lambda) \in l_1$ there is assigned a function $A(t) \in \Pi_W$ as the *generalized discrete Fourier transform* (GDF). Equality (1.2) defines the inverse transform. The sequence is the original $a(\lambda)$, and the function $A(t)$ is the image. The GDF will be denoted by $A(t) = W_0 a(\lambda)$, $a(\lambda) = W_0^{-1} A(t)$. It is proved that if $A(t)$ is differentiable and $A^{(j)}(t) \in \Pi_W$, $j = 0, K, p$, then

$$W_0^{-1} \frac{d^p A(t)}{dt^p} = (i\lambda)^p a(\lambda).$$

The coefficients of the sequence $a(\lambda)$ may depend on y:

$$a(\lambda, y) = \{(a_1, \lambda_1), (a_2, \lambda_2), K\}, \quad y \in [a, b].$$

If there exists a sequence of positive numbers $\{a_n\} \in l_1$ such that $|a_n(y)| \leq \alpha_n$ then the functions $A(t, y) = W_0 a(\lambda, y)$ belong to Π_W on each horizontal straight strip $a \leq \operatorname{Im} z \leq b$ $(z = t + iy)$. We will say that $A(t, y)$ *belongs to* Π_W^y in the strip $[a, b]$. If $A(t, y)$ is differentiable p times with respect to y and $\frac{\partial^j A(t,y)}{\partial y^j} \in \Pi_W^y$, $j = 0, \ldots, p$, then

$$W_0^{-1} \frac{\partial^p A(t, y)}{\partial y^p} = \frac{\partial^p a(\lambda, y)}{\partial y^p}.$$

We will assume that the functions in the boundary conditions belong to Π_W; i.e., are representable as absolutely convergent series

$$\Phi_{km}(x) = \sum_{\lambda} \varphi_{km}(\lambda) e^{i\lambda x}, \quad \Psi_{km}(x) = \sum_{\lambda} \psi_{km}(\lambda) e^{i\lambda x}, \quad k = \overline{1, n}, \ m = \overline{1, l},$$

all functions $F_{jm}(x, y)$ belong to Π_W^y, i.e., are representable as series

$$F_{jm}(x, y) = \sum_{\lambda} f_{jm}(\lambda, y) e^{i\lambda x}, \quad j = \overline{1, n}, \ m = \overline{1, l}.$$

Search for a solution $u_{km}(x, y)$ to the system in the class Π_W^y, i.e., in the form

$$u_{km}(x, y) = \sum_{\lambda} A_{km}(\lambda, y) e^{i\lambda x}, \quad k = \overline{1, n},\ m = \overline{1, l},$$

where $A_{km}(\lambda, y)$ are unknown functions found from the boundary conditions and the gluing conditions as follows:

Apply the operator W_0^{-1} on the equations of (1) and obtain the system

$$\sum_{k=1}^{n} \Bigg(-\lambda^2 a_{kj}^{(m)} A_{km}(\lambda, y) + b_{kj}^{(m)} \frac{d^2 A_{km}(\lambda, y)}{dy^2} + c_{kj}^{(m)} \cdot i\lambda \cdot A_{km}(\lambda, y)$$
$$+ d_{kj}^{(m)} \frac{dA_{km}(\lambda, y)}{dy} + e_{kj}^{(m)} A_{km}(\lambda, y) \Bigg) = f_{jm}(\lambda, y), \quad j = \overline{1, n},\ m = \overline{1, l}, \tag{2}$$

of ordinary differential equations for each fixed m of order at most $2n$ (λ is a parameter). Solving this system, we get

$$A_{km}(\lambda, y) = \sum_{q=1}^{2n} p_{qkm}(\lambda) \xi_{qkm}(\lambda, y) + \tilde{\xi}_{km}(\lambda, y), \quad k = \overline{1, n},\ m = \overline{1, l},$$

where $p_{qkm(\lambda)}$ are constant for fixed q, k, m, and λ. It is these constants that are found from the boundary and gluing conditions. For finding them, we get a system of linear algebraic equations.

As an example, we consider the following problem from filtration theory:

Statement of the problem. In a homogeneous isotropic porous domain consisting of two strips, $-\infty < x < +\infty$: the first ($m = 1$), $-1 \le y \le 0$; the second ($m = 2$), $0 \le y \le 1$, there happens a stationary filtration of some fluid.

On the exterior boundaries of the domain $y = 1$, $y = -1$, the values of the normal and tangential stresses are given, and also the potential of the filtration rate:

$$\begin{aligned} \sigma_y^{(1)}(x, -1) &= F_1(x), \quad & \sigma_y^{(2)}(x, 1) &= G_1(x), \\ \tau_{xy}^{(1)}(x, -1) &= F_2(x), \quad & \tau_{xy}^{(2)}(x, 1) &= G_2(x), \\ \varphi^{(1)}(x, -1) &= F_3(x), \quad & \varphi^{(2)}(x, 1) &= G_3(x), \end{aligned} \tag{3}$$

where $F_j(x)$ and $G_j(x)$, $j = 1, 2, 3$, are almost periodic functions with absolutely convergent Fourier series (belong to the class Π_W), i.e., have the structure

$$F_j(x) = \sum_{\lambda \ne 0} f_j(\lambda) e^{i\lambda x}, \quad G_j(x) = \sum_{\lambda \ne 0} g_j(\lambda) e^{i\lambda x},$$

$\{\lambda\}$ is a countable set of reals separated from zero. On the common boundary of the separation of the media $y = 0$, there hold the rigid attachment conditions:

$$\sigma_y^{(1)}(x, 0) = \sigma_y^{(2)}(x, 0), \quad \tau_{xy}^{(1)}(x, 0) = \tau_{xy}^{(2)}(x, 0), \tag{4}$$

$$u^{(1)}(x, 0) = u^{(2)}(x, 0), \quad \nu^{(1)}(x, 0) = \nu^{(2)}(x, 0), \tag{5}$$

$$\frac{\varphi^{(1)}(x, 0)}{k_1} = \frac{\varphi^{(2)}(x, 0)}{k_2}, \quad \frac{\partial \varphi^{(1)}}{\partial y}(x, 0) = \frac{\partial \varphi^{(2)}}{\partial y}(x, 0), \tag{6}$$

where k_1 and k_2 are the filtration coefficients of the media. The functions $u^{(m)}(x, y)$ and $\nu^{(m)}(x, y)$, $m = 1, 2$, are expressed in terms of $\sigma_x^{(m)}(x, y)$, $\sigma_y^{(m)}(x, y)$, and $\tau_{xy}^{(m)}(x, y)$ in a familiar way.

Find the functions of the potential of the filtration rate $\varphi^{(m)}(x,y)$ of the fluid acting in each of the stripes of the porous domain $0 \le y \le 1$, $-1 \le y \le 0$, $-\infty < x < +\infty$, and also the stresses $\sigma_x^{(m)}(x,y)$, $\sigma_y^{(m)}(x,y)$, $\tau_{xy}^{(m)}(x,y)$, $m = 1,2$.

Such problems appear in calculating the stress and deformation of pavements [4].

This problem is reduced to solving a system of equations with respect to $\sigma_x^{(m)}(x,y)$, $\sigma_y^{(m)}(x,y)$, $\tau_{xy}^{(m)}(x,y)$, $\varphi^{(m)}(x,y)$, $m = 1,2$ [5]:

$$\begin{aligned} \frac{\partial^2 \sigma_x^{(m)}}{\partial x^2} + \frac{\partial^2 \sigma_y^{(m)}}{\partial x^2} + \frac{\partial^2 \sigma_y^{(m)}}{\partial x^2} + \frac{\partial^2 \sigma_y^{(m)}}{\partial y^2} &= 0, \\ \frac{\partial \sigma_x^{(m)}}{\partial x} + \frac{\partial \tau_{xy}^{(m)}}{\partial y} - \frac{\partial \varphi^{(m)}}{\partial x} &= 0, \\ \frac{\partial \tau_{xy}^{(m)}}{\partial x} + \frac{\partial \sigma_y^{(m)}}{\partial y} - \frac{\partial \varphi^{(m)}}{\partial y} - k_0^{(m)} &= 0, \\ \frac{\partial^2 \varphi^{(m)}}{\partial x^2} + \frac{\partial^2 \varphi^{(m)}}{\partial y^2} &= 0. \end{aligned} \tag{7}$$

Search for a solution in the class Π_W^y $(0 \le y \le 1)$, i.e., in the class of functions representable as series

$$\begin{aligned} \sigma_x^{(1)}(x,y) &= \sum_\lambda A_\lambda^{(1)}(y) e^{i\lambda x}, & \sigma_x^{(2)}(x,y) &= \sum_\lambda A_\lambda^{(2)}(y) e^{i\lambda x}, \\ \sigma_y^{(1)}(x,y) &= \sum_\lambda B_\lambda^{(1)}(y) e^{i\lambda x}, & \sigma_y^{(2)}(x,y) &= \sum_\lambda B_\lambda^{(2)}(y) e^{i\lambda x}, \\ \tau_{xy}^{(1)}(x,y) &= \sum_\lambda C_\lambda^{(1)}(y) e^{i\lambda x}, & \tau_{xy}^{(2)}(x,y) &= \sum_\lambda C_\lambda^{(2)}(y) e^{i\lambda x}, \\ \varphi^{(1)}(x,y) &= \sum_\lambda D_\lambda^{(1)}(y) e^{i\lambda x}, & \varphi^{(2)}(x,y) &= \sum_\lambda D_\lambda^{(2)}(y) e^{i\lambda x}. \end{aligned} \tag{8}$$

Apply the operator W_0^{-1} at the equations of (7). Involving the properties of W_0^{-1}, we arrive at the equations

$$\begin{aligned} -\lambda^2 A_\lambda^{(m)}(y) + \frac{d^2 A_\lambda^{(m)}(y)}{dy^2} - \lambda^2 B_\lambda^{(m)}(y) + \frac{d^2 B_\lambda^{(m)}(y)}{dy^2} &= 0, \\ i\lambda A_\lambda^{(m)}(y) + \frac{dC_\lambda^{(m)}(y)}{dy} - i\lambda D_\lambda^{(m)}(y) &= 0, \\ i\lambda C_\lambda^{(m)}(y) + \frac{dB_\lambda^{(m)}(y)}{dy} - \frac{D_\lambda^{(m)}(y)}{dy} - k_0^{(m)} &= 0, \\ -\lambda^2 D_\lambda^{(m)}(y) + \frac{d^2 D_\lambda^{(m)}(y)}{dy^2} &= 0, \quad m = 1,2. \end{aligned} \tag{9}$$

Solving the system of differential equations (9), we infer

$$
\begin{aligned}
A_\lambda^{(m)}(y) &= \left(-b_1^{(m)}(\lambda) - \frac{2}{\lambda} b_3^{(m)}(\lambda) - b_3^{(m)}(\lambda) y + 2d_1^{(m)}(\lambda)\right) e^{\lambda y} \\
&\quad + \left(-b_2^{(m)}(\lambda) + \frac{2}{\lambda} b_4^{(m)}(\lambda) - b_4^{(m)}(\lambda) y + 2d_2^{(m)}(\lambda)\right) e^{-\lambda y}, \\
B_\lambda^{(m)}(y) &= \left(b_1^{(m)}(\lambda) + b_3^{(m)}(\lambda) y\right) e^{\lambda y} + \left(b_2^{(m)}(\lambda) + b_4^{(m)}(\lambda) y\right) e^{-\lambda y}, \\
C_\lambda^{(m)}(y) &= -\frac{ik_0^{(m)}}{\lambda} + i\left(b_1^{(m)}(\lambda) + \frac{1}{\lambda} b_3^{(m)}(\lambda) + b_3^{(m)}(\lambda) y - d_1^{(m)}(\lambda)\right) e^{\lambda y} \\
&\quad + i\left(-b_2^{(m)}(\lambda) + \frac{1}{\lambda} b_4^{(m)}(\lambda) - b_4^{(m)}(\lambda) y + d_2^{(m)}(\lambda)\right) e^{-\lambda y}, \\
D_\lambda^{(m)}(y) &= d_1^{(m)}(\lambda) e^{\lambda y} + d_2^{(m)}(\lambda) e^{-\lambda y}, \quad m = 1, 2.
\end{aligned}
\tag{10}
$$

Use the boundary and attachment conditions for finding $b_1^{(m)}(\lambda)$, $b_2^{(m)}(\lambda)$, $b_3^{(m)}(\lambda)$, $b_4^{(m)}(\lambda)$, $d_1^{(m)}(\lambda)$, and $d_2^{(m)}(\lambda)$, $m = 1, 2$. The boundary conditions (3) imply that

$$b_1^{(1)}(\lambda)e^{-\lambda} + b_2^{(1)}(\lambda)e^{\lambda} - b_3^{(1)}(\lambda)e^{-\lambda} - b_4^{(1)}(\lambda)e^{\lambda} = f_1(\lambda),$$

$$
\begin{aligned}
&-\frac{ik_0^{(1)}}{\lambda} + i\left(b_1^{(1)}(\lambda) + \frac{1}{\lambda} b_3^{(1)}(\lambda) - b_3^{(1)}(\lambda) - d_1^{(1)}(\lambda)\right) e^{-\lambda} \\
&+i\left(-b_2^{(1)}(\lambda) + \frac{1}{\lambda} b_4^{(1)}(\lambda) + b_4^{(1)}(\lambda) + d_2^{(1)}(\lambda)\right) e^{\lambda} = f_2(\lambda),
\end{aligned}
$$

$$d_1^{(1)}(\lambda)e^{-\lambda} + d_2^{(1)}(\lambda)e^{\lambda} = f_3(\lambda),$$

$$b_1^{(2)}(\lambda)e^{\lambda} + b_2^{(2)}(\lambda)e^{-\lambda} + b_3^{(2)}(\lambda)e^{\lambda} + b_4^{(2)}(\lambda)e^{-\lambda} = g_1(\lambda),$$

$$
\begin{aligned}
&-\frac{ik_0^{(2)}}{\lambda} + i\left(b_1^{(2)}(\lambda) + \frac{1}{\lambda} b_3^{(2)}(\lambda) + b_3^{(2)}(\lambda) - d_1^{(2)}(\lambda)\right) e^{\lambda} \\
&+i\left(-b_2^{(2)}(\lambda) + \frac{1}{\lambda} b_4^{(2)}(\lambda) - b_4^{(2)}(\lambda) + d_2^{(2)}(\lambda)\right) e^{-\lambda} = g_2(\lambda),
\end{aligned}
$$

$$d_1^{(2)}(\lambda)e^{\lambda} + d_2^{(2)}(\lambda)e^{-\lambda} = g_3(\lambda).$$

Conditions (4) imply that

$$b_1^{(1)}(\lambda) + b_2^{(1)}(\lambda) = b_1^{(2)}(\lambda) + b_2^{(2)}(\lambda),$$

$$
\begin{aligned}
&-\frac{k_0^{(1)}}{\lambda} + \left(b_1^{(1)}(\lambda) + \frac{1}{\lambda} b_3^{(1)}(\lambda) - d_1^{(1)}(\lambda)\right) + \left(-b_2^{(1)}(\lambda) + \frac{1}{\lambda} b_4^{(1)}(\lambda) + d_2^{(1)}(\lambda)\right) \\
&= -\frac{k_0^{(2)}}{\lambda} + \left(b_1^{(2)}(\lambda) + \frac{1}{\lambda} b_3^{(2)}(\lambda) - d_1^{(2)}(\lambda)\right) + \left(-b_2^{(2)}(\lambda) + \frac{1}{\lambda} b_4^{(2)}(\lambda) + d_2^{(2)}(\lambda)\right).
\end{aligned}
$$

The functions $u^{(m)}(x,y)$ and $\nu^{(m)}(x,y)$, $m=1,2$, are determined as follows:

$$u^{(m)}(x,y)$$
$$= \sum_{\lambda\neq 0} \frac{i}{E_m\lambda^2}\big(\big(b_1^{(m)}\lambda + 2b_3^{(m)} + \lambda y b_3^{(m)} - 2\lambda d_1^{(m)} + \nu_m\lambda b_1^{(m)} + \nu_m\lambda b_3^{(m)} y\big)e^{\lambda y}$$
$$+\big(\lambda b_2^{(m)} - 2b_4^{(m)} + \lambda y b_4^{(m)} - 2\lambda d_2^{(m)} + \nu_m\lambda b_2^{(m)} - \nu_m\lambda b_4^{(m)} y\big)e^{-\lambda y}\big)e^{i\lambda x},$$

$$\nu^{(m)}(x,y) = \sum_{\lambda\neq 0} \frac{i}{E_m\lambda^2}\big(\big(b_1^{(m)}\lambda - b_3^{(m)} + \lambda y b_3^{(m)} - 2\nu_m\lambda d_1^{(m)} + \nu_m\lambda b_1^{(m)}$$
$$+\nu_m b_3^{(m)} + \nu_m\lambda b_3^{(m)} y\big)e^{\lambda y} + (-\lambda b_2^{(m)} - b_4^{(m)} - \lambda y b_4^{(m)} + 2\nu_m\lambda d_2^{(m)}$$
$$-\nu_m\lambda b_2^{(m)} + \nu_m b_4^{(m)} - \nu_m\lambda b_4^{(m)} y)e^{-\lambda y}\big)e^{i\lambda x} + \sum_{\lambda\neq 0}\frac{-2k_0^{(m)}(1+\nu_m)}{\lambda^2 E_m}e^{i\lambda x},$$

where E_m and ν_m are given constants. Conditions (5) imply

$$\sum_{\lambda\neq 0} \frac{i}{E_1\lambda^2}\big(\big(b_1^{(1)}\lambda + 2b_3^{(1)} - 2\lambda d_1^{(1)} + \nu_1\lambda b_1^{(1)}\big)$$
$$+\big(\lambda b_2^{(1)} - 2b_4^{(1)} - 2\lambda d_2^{(1)} + \nu_1\lambda b_2^{(1)}\big)\big)e^{i\lambda x}$$
$$= \sum_{\lambda\neq 0} \frac{i}{E_2\lambda^2}\big(\big(b_1^{(2)}\lambda + 2b_3^{(2)} - 2\lambda d_1^{(2)} + \nu_2\lambda b_1^{(2)}\big)$$
$$+\big(\lambda b_2^{(2)} - 2b_4^{(2)} - 2\lambda d_2^{(2)} + \nu_2\lambda b_2^{(2)}\big)\big)e^{i\lambda x},$$

$$\sum_{\lambda\neq 0} \frac{i}{E_1\lambda^2}\big(\big(b_1^{(1)}\lambda - b_3^{(1)} - 2\nu_1\lambda d_1^{(1)} + \nu_1\lambda b_1^{(1)} + \nu_1 b_3^{(1)}\big)$$
$$+\big(-\lambda b_2^{(1)} - b_4^{(1)} - 2\nu_1\lambda d_2^{(1)} - \nu_1\lambda b_2^{(1)} + \nu_1 b_4^{(1)}\big) - 2k_0^{(1)}(1+\nu_1)\big)e^{i\lambda x}$$
$$= \sum_{\lambda\neq 0} \frac{i}{E_2\lambda^2}\big(\big(b_1^{(2)}\lambda - 2b_3^{(2)} - 2\nu_2\lambda d_1^{(2)} + \nu_2\lambda b_1^{(2)} + \nu_2 b_3^{(2)}\big)$$
$$+\big(-\lambda b_2^{(2)} - b_4^{(2)} + 2\nu_2\lambda d_2^{(2)} - \nu_2\lambda b_2^{(2)} + \nu_2 b_4^{(2)}\big) - 2k_0^{(2)}(1+\nu_2)\big)e^{i\lambda x},$$

It follows from (6) that

$$\frac{d_1^{(1)}(\lambda) + d_2^{(1)}(\lambda)}{k_1} = \frac{d_1^{(2)}(\lambda) + d_2^{(2)}(\lambda)}{k_2}, \quad d_1^{(1)}(\lambda) - d_2^{(1)}(\lambda) = d_1^{(2)}(\lambda) - d_2^{(2)}(\lambda).$$

By what was stated above, for finding the coefficients $b_1^{(m)}(\lambda)$, $b_2^{(m)}(\lambda)$, $b_3^{(m)}(\lambda)$, $b_4^{(m)}(\lambda)$, $d_1^{(m)}(\lambda)$, and $d_2^{(m)}(\lambda)$, $m=1,2$, we must solve the system of linear algebraic equations

$$
\begin{aligned}
& e^{-\lambda}b_1^{(1)}(\lambda)+e^{\lambda}b_2^{(1)}(\lambda)-e^{-\lambda}b_3^{(1)}(\lambda)-e^{\lambda}b_4^{(1)}(\lambda)=f_1(\lambda),\\
& e^{-\lambda}b_1^{(1)}(\lambda)-e^{\lambda}b_2^{(1)}(\lambda)+\left(\frac{1}{\lambda}-1\right)e^{-\lambda}b_3^{(1)}(\lambda)+\left(\frac{1}{\lambda}+1\right)e^{\lambda}b_4^{(1)}(\lambda)\\
&\qquad -e^{-\lambda}d_1^{(1)}(\lambda)+e^{\lambda}d_2^{(1)}(\lambda)=-if_2(\lambda)+\frac{k_0^{(1)}}{\lambda},\\
& e^{-\lambda}d_1^{(1)}(\lambda)+e^{\lambda}d_2^{(1)}(\lambda)=f_3(\lambda),\\
& e^{\lambda}b_1^{(2)}(\lambda)+e^{-\lambda}b_2^{(2)}(\lambda)+e^{\lambda}b_3^{(2)}(\lambda)+e^{-\lambda}b_4^{(2)}(\lambda)=g_1(\lambda),\\
& e^{\lambda}b_1^{(2)}(\lambda)-e^{-\lambda}b_2^{(2)}(\lambda)+\left(\frac{1}{\lambda}+1\right)e^{\lambda}b_3^{(2)}(\lambda)+\left(\frac{1}{\lambda}-1\right)e^{-\lambda}b_4^{(2)}(\lambda)\\
&\qquad -e^{\lambda}d_1^{(2)}(\lambda)+e^{-\lambda}d_2^{(2)}(\lambda)=-ig_2(\lambda)+\frac{k_0^{(2)}}{\lambda},\\
& e^{\lambda}d_1^{(2)}(\lambda)+e^{-\lambda}d_2^{(2)}(\lambda)=g_3(\lambda),\\
& b_1^{(1)}(\lambda)+b_2^{(1)}(\lambda)-b_1^{(2)}(\lambda)-b_2^{(2)}(\lambda)=0,\\
& b_1^{(1)}(\lambda)-b_2^{(1)}(\lambda)+\frac{1}{\lambda}b_3^{(1)}(\lambda)+\frac{1}{\lambda}b_4^{(1)}(\lambda)-d_1^{(1)}(\lambda)+d_2^{(1)}(\lambda)-b_1^{(2)}(\lambda)\\
&\qquad +b_2^{(2)}(\lambda)-\frac{1}{\lambda}b_3^{(2)}(\lambda)-\frac{1}{\lambda}b_4^{(2)}(\lambda)+d_1^{(2)}(\lambda)-d_2^{(2)}(\lambda)=\frac{k_0^{(1)}}{\lambda}-\frac{k_0^{(2)}}{\lambda},\\
& k_2d_1^{(1)}(\lambda)+k_2d_2^{(1)}(\lambda)-k_1d_1^{(2)}(\lambda)-k_1d_2^{(2)}(\lambda)=0,\\
& d_1^{(1)}(\lambda)-d_2^{(1)}(\lambda)-d_1^{(2)}(\lambda)+d_2^{(2)}(\lambda)=0,\\
& \frac{\lambda(1+\nu_1)}{E_1}b_1^{(1)}+\frac{\lambda(1+\nu_1)}{E_1}b_2^{(1)}+\frac{2}{E_1}b_3^{(1)}(\lambda)-\frac{2}{E_1}b_4^{(1)}(\lambda)-\frac{2\lambda}{E_1}d_1^{(1)}-\frac{2\lambda}{E_1}d_2^{(1)}\\
&\qquad -\frac{\lambda(1+\nu_2)}{E_2}b_1^{(2)}-\frac{\lambda(1+\nu_2)}{E_2}b_2^{(2)}-\frac{2}{E_2}b_3^{(2)}(\lambda)\\
&\qquad +\frac{2}{E_2}b_4^{(2)}(\lambda)+\frac{2\lambda}{E_2}d_1^{(2)}+\frac{2\lambda}{E_2}d_2^{(2)}=0,\\
& \frac{\lambda(1+\nu_1)}{E_1}b_1^{(1)}-\frac{\lambda(1+\nu_1)}{E_1}b_2^{(1)}+\frac{\nu_1-1}{E_1}b_3^{(1)}(\lambda)+\frac{\nu_1-1}{E_1}b_4^{(1)}(\lambda)-\frac{2\lambda\nu_1}{E_1}d_1^{(1)}\\
&\qquad +\frac{2\lambda\nu_1}{E_1}d_2^{(1)}-\frac{\lambda(1+\nu_2)}{E_2}b_1^{(2)}+\frac{\lambda(1+\nu_2)}{E_2}b_2^{(2)}-\frac{(\nu_2-1)}{E_2}b_3^{(2)}(\lambda)\\
&\qquad -\frac{(\nu_2-1)}{E_2}b_4^{(2)}(\lambda)+\frac{2\lambda\nu_2}{E_2}d_1^{(2)}-\frac{2\lambda\nu_2}{E_2}d_2^{(2)}\\
&\qquad =\frac{2k_0^{(1)}(1+\nu_1)}{E_1}-\frac{2k_0^{(2)}(1+\nu_2)}{E_2}.
\end{aligned}
\tag{11}
$$

This system splits into two subsystems

$$
\begin{aligned}
e^{-\lambda}d_1^{(1)}(\lambda)+e^{\lambda}d_2^{(1)}(\lambda)&=f_3(\lambda),\\
e^{\lambda}d_1^{(2)}(\lambda)+e^{-\lambda}d_2^{(2)}(\lambda)&=g_3(\lambda),\\
k_2d_1^{(1)}(\lambda)+k_2d_2^{(1)}(\lambda)-k_1d_1^{(2)}(\lambda)-k_1d_2^{(2)}(\lambda)&=0,\\
d_1^{(1)}(\lambda)-d_2^{(1)}(\lambda)-d_1^{(2)}(\lambda)+d_2^{(2)}(\lambda)&=0,
\end{aligned}
\tag{12}
$$

and

$$e^{-\lambda}b_1^{(1)}(\lambda)+e^{\lambda}b_2^{(1)}(\lambda)-e^{-\lambda}b_3^{(1)}(\lambda)-e^{\lambda}b_4^{(1)}(\lambda)=f_1(\lambda),$$

$$e^{-\lambda}b_1^{(1)}(\lambda)-e^{\lambda}b_2^{(1)}(\lambda)+\left(\frac{1}{\lambda}-1\right)e^{-\lambda}b_3^{(1)}(\lambda)+\left(\frac{1}{\lambda}+1\right)e^{\lambda}b_4^{(1)}(\lambda)$$

$$=-if_2(\lambda)+\frac{k_0^{(1)}}{\lambda}+e^{-\lambda}d_1^{(1)}(\lambda)-e^{\lambda}d_2^{(1)}(\lambda),$$

$$e^{\lambda}b_1^{(2)}(\lambda)+e^{-\lambda}b_2^{(2)}(\lambda)+e^{\lambda}b_3^{(2)}(\lambda)+e^{-\lambda}b_4^{(2)}(\lambda)=g_1(\lambda),$$

$$e^{\lambda}b_1^{(2)}(\lambda)-e^{-\lambda}b_2^{(2)}(\lambda)+\left(\frac{1}{\lambda}+1\right)e^{\lambda}b_3^{(2)}(\lambda)+\left(\frac{1}{\lambda}-1\right)e^{-\lambda}b_4^{(2)}(\lambda)$$

$$=-ig_2(\lambda)+\frac{k_0^{(2)}}{\lambda}+e^{\lambda}d_1^{(2)}(\lambda)-e^{-\lambda}d_2^{(2)}(\lambda),$$

$$b_1^{(1)}(\lambda)+b_2^{(1)}(\lambda)-b_1^{(2)}(\lambda)-b_2^{(2)}(\lambda)=0, \tag{13}$$

$$b_1^{(1)}(\lambda)-b_2^{(1)}(\lambda)+\frac{1}{\lambda}b_3^{(1)}(\lambda)+\frac{1}{\lambda}b_4^{(1)}(\lambda)-b_1^{(2)}(\lambda)+b_2^{(2)}(\lambda)$$

$$-\frac{1}{\lambda}b_3^{(2)}(\lambda)-\frac{1}{\lambda}b_4^{(2)}(\lambda)=\frac{k_0^{(1)}}{\lambda}-\frac{k_0^{(2)}}{\lambda}+d_1^{(1)}(\lambda)-d_2^{(1)}(\lambda)-d_1^{(2)}(\lambda)+d_2^{(2)}(\lambda),$$

$$\frac{\lambda(1+\nu_1)}{E_1}b_1^{(1)}+\frac{\lambda(1+\nu_1)}{E_1}b_2^{(1)}+\frac{2}{E_1}b_3^{(1)}(\lambda)-\frac{2}{E_1}b_4^{(1)}(\lambda)-\frac{\lambda(1+\nu_2)}{E_2}b_1^{(2)}$$

$$-\frac{\lambda(1+\nu_2)}{E_2}b_2^{(2)}-\frac{2}{E_2}b_3^{(2)}(\lambda)+\frac{2}{E_2}b_4^{(2)}(\lambda)=\frac{2\lambda}{E_1}d_1^{(1)}+\frac{2\lambda}{E_1}d_2^{(1)}-\frac{2\lambda}{E_2}d_1^{(2)}-\frac{2\lambda}{E_2}d_2^{(2)}=0,$$

$$\frac{\lambda(1+\nu_1)}{E_1}b_1^{(1)}-\frac{\lambda(1+\nu_1)}{E_1}b_2^{(1)}+\frac{\nu_1-1}{E_1}b_3^{(1)}(\lambda)+\frac{\nu_1-1}{E_1}b_4^{(1)}(\lambda)$$

$$-\frac{\lambda(1+\nu_2)}{E_2}b_1^{(2)}+\frac{\lambda(1+\nu_2)}{E_2}b_2^{(2)}-\frac{(\nu_2-1)}{E_2}b_3^{(2)}(\lambda)-\frac{(\nu_2-1)}{E_2}b_4^{(2)}(\lambda)$$

$$=\frac{2k_0^{(1)}(1+\nu_1)}{E_1}-\frac{2k_0^{(2)}(1+\nu_2)}{E_2}+\frac{2\lambda\nu_1}{E_1}d_1^{(1)}-\frac{2\lambda\nu_1}{E_1}d_2^{(1)}-\frac{2\lambda\nu_2}{E_2}d_1^{(2)}+\frac{2\lambda\nu_2}{E_2}d_2^{(2)}.$$

The determinant of each of the two systems is nonzero for all $\lambda\neq 0$. Thus, we have

Theorem. *Suppose that the functions F_j and G_j, $j=1,2,3$, belong to Π_W, i.e., have the form*

$$F_j=\sum_{\lambda\neq 0}f_j(\lambda)e^{i\lambda x},\quad G_j=\sum_{\lambda\neq 0}g_j(\lambda)e^{i\lambda x},$$

where the set $\{\lambda\}$ is separated from zero. Then, in the domain consisting of the two strips $0\le y\le 1$, $-1\le y\le 0$, $-\infty<x<+\infty$, the boundary value problem (3)–(7) has a unique solution, which is represented by series (8), where $A_\lambda^{(m)}(y)$, $B_\lambda^{(m)}(y)$, $C_\lambda^{(m)}(y)$, $D_\lambda^{(m)}(y)$ are found from (10) and the coefficients $b_1^{(m)}(\lambda)$, $b_2^{(m)}(\lambda)$, $b_3^{(m)}(\lambda)$, $b_4^{(m)}(\lambda)$, $d_1^{(m)}(\lambda)$, and $d_2^{(m)}(\lambda)$, $m=1,2$, are found from (12), (13). All series converge absolutely and uniformly over x provided that the series $\sum\frac{1}{|\lambda|}$ converges.

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M. F. Kulagina; E. A. Mikishanina
I. N. Ul′yanov Chuvash State University, Cheboksary, Russia
kulagina_mf@mail.ru; evaeva_84@mail.ru

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SOLVABILITY OF AN INVERSE COEFFICIENT PROBLEM FOR A NONCLASSICAL EQUATION OF THIRD ORDER

N. N. Nikolaev and S. V. Popov

Abstract. An inverse problem of recovering time-dependent external sources together with a solution with point overdetermination conditions is considered for a third order equation and densities of one or two sources are recovered. Existence and uniqueness of solutions to a coefficient inverse problem are proven.

Keywords: coefficient inverse problem, third order equation, overdetermination condition, existence of a solution, uniqueness, Sobolev space, method of continuation in a parameter, regularization, a priori estimate

1. Introduction

The problems of recovering coefficients of partial differential equations and systems with given additional information about a solution are of great practical importance [1–3]. Note that the inverse problems for hyperbolic equations often are regarded as ill-posed problems of mathematical physics whose theory was founded in articles by A. N. Tikhonov [4–6], V. K. Ivanov [7], and M. M. Lavrent′ev [8, 9].

The problems of recovering densities of external sources often arise in the theory of inverse problems of heat and mass transfer. It is often the case when the unknown right-hand side depends on time [13] and inverse problems are stated as control problems [14]. The articles [15, 16] are devoted to the study of inverse problems for higher order parabolic equations. Observe that direct spatially nonlocal problems for third order equations are well studied (see, for instance, [17–19]) in contrast to inverse problems for equations of this type. The unknown parameter depending on time is examined in [20, 21] for parabolic equations and in [22–24] for hyperbolic.

In this article we establish solvability of an inverse problem of recovering external sources together with a solution for a third order equation in time with point overdetermination conditions; the densities of one or two sources are recovered.

2. Statements of Inverse Boundary Value Problems

Assume that Ω is the interval $(0, 1)$ of the Ox-axis and Q is the rectangle $\Omega \times (0, T)$ with $0 < T < +\infty$.

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Boundary Value Problem 1. *Find $u(x,t)$ and $q(t)$ satisfying*

$$u_{ttt} + u_{xx} + c(x,t)u = f(x,t) + q(t)h(x,t) \tag{1}$$

in Q, the initial conditions

$$u(x,0) = u_t(x,0) = u(x,T) = 0, \quad x \in \Omega, \tag{2}$$

the boundary conditions

$$u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t \in (0,T), \tag{3}$$

and the overdetermination conditions

$$u(0,t) = 0, \quad t \in (0,T). \tag{4}$$

Boundary Value Problem 2. *Find $u(x,t)$, $q_1(t)$, and $q_2(t)$ satisfying the equation*

$$u_{ttt} + u_{xx} + c(x,t)u = f(x,t) + q_1(t)h_1(x,t) + q_2(t)h_2(x,t), \tag{5}$$

in Q, the initial conditions (2)*, the boundary conditions* (3)*, and the overdetermination conditions*

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t \in (0,T). \tag{6}$$

The Fourier method is applied to study an inverse problem of recovering densities of sources in the one-dimensional wave equation with constant coefficients in [22].

3. Solvability of Boundary Value Problem 1

To simplify exposition, we put $c(x,t) = c(t)$ and introduce the notations

$$f_1(x,t) = f(x,t) - \frac{f(0,t)h(x,t)}{h(0,t)}, \quad h_1(x,t) = \frac{h(x,t)}{h(0,t)},$$

$$\alpha_0(t) = f_{1x}(0,t), \quad \alpha_1(t) = h_{1x}(0,t), \quad \beta_0(t) = f_{1x}(1,t), \quad \beta_1(t) = h_{1x}(1,t).$$

Put

$$h_0 = \max_{\overline{Q}} |h_{1xx}|. \tag{7}$$

Assume that $V_0 = W^{2,3}_{2,x,t}(Q)$ is an anisotropic Sobolev space and W_0 and W_1 are the vector spaces

$$W_0 = \{v(x,t) : v(x,t) \in V_0,\ v_{xxttt} \in L_2(Q)\},$$

$$W_1 = \{v(x,t) : v(x,t) \in W_0,\ v_x \in W_0\}$$

endowed with the norms $\|v\|_{W_0} = \|v\|_{V_0} + \|v_{xxttt}\|_{L_2(Q)}$ and $\|v\|_{W_1} = \|v\|_{W_0} + \|v_x\|_{W_0}$.

Before proving solvability of Boundary Value Problem 1, we observe that $v(x,t)$ from V_0 satisfying (2) meet the inequalities

$$\begin{aligned} v^2(0,t) &\le \delta_1 \int_0^1 v_x^2(x,t)\,dx + C_1(\delta_1) \int_0^1 v^2(x,t)\,dx, \\ v^2(1,t) &\le \delta_1 \int_0^1 v_x^2(x,t)\,dx + C_1(\delta_1) \int_0^1 v^2(x,t)\,dx; \end{aligned} \tag{8}$$

$$\int_0^T v_t^2(x,t)\,dt \le \delta_2 \int_0^T v_{ttt}^2(x,t)\,dt + C_2(\delta_2,T)\int_0^T v^2(x,t)\,dt; \tag{9}$$

$$\int_0^T v_{tt}^2(x,t)\,dt \le \delta_3 \int_0^T v_{ttt}^2(x,t)\,dt + C_3(\delta_3,T)\int_0^T v^2(x,t)\,dt, \tag{10}$$

where δ_1, δ_2, and δ_3 are arbitrary positive numbers and C_1, C_2, and C_3 are calculated through δ_1, δ_2, δ_3, and T.

Theorem 1. *Assume that*

$$c(t) \in C^1[0,T], \quad -c(t) \ge c_0 \gg 0 \quad \textit{for} \quad t \in [0,T], \tag{11}$$

$$h(x,t) \in C^3(\overline{Q}), \quad h_0 < \frac{1}{2T}, \quad h(0,t) \ne 0, \quad \alpha_1^2(t) + \beta_1^2(t) \le \frac{1}{2} \quad \textit{for} \quad t \in [0,T], \tag{12}$$

$$\alpha_1(t)\xi_1^2 - \beta_1(t)\xi_1\xi_2 + \frac{1}{8}\xi_2^2 \ge 0 \quad \textit{for} \quad t \in [0,T], \quad (\xi_1,\xi_2) \in \mathbb{R}^2, \tag{13}$$

$$f(x,t) \in W_2^3(Q), \quad f_{xxttt}(x,t) \in L_2(Q), \tag{14}$$

$$\begin{gathered} f_x(0,0) = h_x(0,0) = f_x(1,0) = h_x(1,0) = 0, \\ f_x(0,T) = h_x(0,T) = f_x(1,T) = h_x(1,T) = 0, \\ f_{xt}(0,0) = h_{xt}(0,0) = f_{xt}(1,0) = h_{xt}(1,0) = 0. \end{gathered} \tag{15}$$

Then there exists a regular solution to (1)–(4) *such that* $u(x,t)$ *and* $u_{xx}(x,t)$ *belong to* $W^{2,3}_{2,x,t}(Q)$ *and* $q(t) \in L_2(0,T)$.

Proof. Consider the auxiliary boundary value problem: *Find a solution* $u(x,t)$ *to*

$$u_{ttt} + u_{xx} + c(t)u = f_{1xx}(x,t) + \lambda h_{1xx}(x,t)u(0,t) \tag{16}$$

in Q *such that the nonlocal conditions*

$$\begin{gathered} u_x(0,t) = \alpha_1(t)u(0,t) + \alpha_0(t), \quad 0 < t < T, \\ u_x(1,t) = \beta_1(t)u(0,t) + \beta_0(t), \quad 0 < t < T, \end{gathered} \tag{17}$$

and the initial conditions

$$u(x,0) = u_t(x,0) = u(x,T) = 0, \quad x \in \Omega, \tag{18}$$

hold, where λ *is some real parameter to be defined below.*

Note that spatially nonlocal boundary value problems of the form (16)–(18) for nonloaded equations (16) (without the factors $h_{1xx}(x,t)u(0,t)$), are considered in [25, 26].

Given $(x,t) \in \overline{Q}$, put

$$\gamma(x,t) = \frac{x^2}{2}[\beta_0(t) - \alpha_0(t)] + x\alpha_0(t), \quad v(x,t) = u(x,t) - \gamma(x,t).$$

In this case, instead of (16)–(18) we can consider the boundary value problem: *Find a solution* $v(x,t)$ *to*

$$v_{ttt} + v_{xx} + c(t)v = f_2(x,t) + \lambda h_{1xx}(x,t)v(0,t) \tag{19}$$

in Q such that

$$v_x(0,t) = \alpha_1(t)v(0,t), \quad 0 < t < T,$$
$$v_x(1,t) = \beta_1(t)v(0,t), \quad 0 < t < T, \tag{20}$$

$$v(x,0) = v_t(x,0) = v(x,T) = 0, \quad x \in \Omega. \tag{21}$$

where

$$f_2(x,t) = f_{1xx}(x,t) + B_0(x,t), \quad B_0(x,t) = -\gamma_{ttt}(x,t) - \gamma_{xx}(x,t) - c(t)\gamma(x,t).$$

Without loss of generality, we can assume that

$$\beta_0(0) = \beta_0'(0) = 0, \quad \alpha_0(0) = \alpha_0'(0) = 0, \quad \alpha_0(T) = \beta_0(T) = 0.$$

These equalities are fulfilled, for example, if (15) holds.

Given $(x,t) \in \overline{Q}$, we assign

$$\delta(x,t,\lambda) = \frac{\lambda x^2}{2}[\beta_1(t) - \alpha_1(t)] + \lambda x\alpha_1(t), \quad w(x,t) = v(x,t) - \delta(x,t,\lambda)v(0,t).$$

The dependence of $w(x,t)$ on λ is omitted for simplicity.

Taking $x = 0$ and $x = 1$ in the equality defining $w(x,t)$, we find that $w(0,t)$ and $w(1,t)$ can be calculated trough $v(0,t)$ and $v(1,t)$ in accord with the formulas

$$w(0,t) = v(0,t), \quad w(1,t) = -\delta(1,t,\lambda)v(0,t) + v(1,t).$$

We have

$$v(x,t) = w(x,t) + \delta(x,t,\lambda)w(0,t).$$

Let $v(x,t)$ be a solution to (19). In this case $w(x,t)$ satisfies the equality

$$w_{ttt} + w_{xx} + c(t)w = f_2(x,t) + \Phi(x,t,\lambda,\overline{w}(t)),$$

where

$$\overline{w}(t) = (w_{ttt}(0,t), w_{tt}(0,t), w_t(0,t), w(0,t)),$$
$$\Phi(x,t,\lambda,\overline{w}(t)) = B_1(x,t,\lambda)w_{ttt}(0,t) + B_2(x,t,\lambda)w_{tt}(0,t)$$
$$+B_3(x,t,\lambda)w_t(0,t) + B_4(x,t,\lambda)w(0,t),$$
$$B_1(x,t,\lambda) = -\delta(x,t,\lambda), \quad B_2(x,t,\lambda) = -3\delta_t(x,t,\lambda), \quad B_3(x,t,\lambda) = -3\delta_{tt}(x,t,\lambda),$$
$$B_4(x,t,\lambda) = \lambda h_{1xx}(x,t) - \delta_{ttt}(x,t,\lambda) - \delta_{xx}(x,t,\lambda) - c(t)\delta(x,t,\lambda).$$

Consider the auxiliary boundary value problem : *Find a solution $w(x,t)$ to*

$$w_{ttt} + w_{xx} + c(t)w = f_2(x,t) + \Phi(x,t,\lambda,\overline{w}(t)) \tag{22}$$

in the rectangle Q such that

$$w_x(0,t) = w_x(1,t) = 0, \quad t \in (0,T),$$
$$w(x,0) = w_t(x,0) = w(x,T) = 0, \quad x \in \Omega. \tag{23}$$

Prove that this problem is solvable in V_0. To this end, we employ the methods of continuation in a parameter and regularization.

Let ε be a positive number; without loss of generality we can assume that $0 < \varepsilon < 1$. Examine the new boundary value problem: *Find a solution $w(x,t)$ to*

$$L_\varepsilon(\lambda)w \equiv w_{ttt} + w_{xx} + c(t)w - \varepsilon w_{xxttt} = f_2(x,t) + \Phi(x,t,\lambda,\overline{w}(t)) \tag{24}$$

in Q such that (23) *hold.*

Demonstrate that the boundary value problem (24), (23) for a fixed parameter $\varepsilon > 0$ is solvable in W_1 for every $f_2(x,t) \in L_2(Q)$ such that $f_{2x}(x,t) \in L_2(Q)$.

In accord with the method of continuation in a parameter [27], for the boundary value problem (24), (23) to be solvable in the space W_1 for all $\lambda \in [0,1]$ and every $f(x,t)$ from $W^{1,0}_{2,x,t}(Q)$, it is sufficient to establish

1) the continuity of the family of operators $\{L_\varepsilon(\lambda)\}$ in λ;

2) solvability of the boundary value problem (24), (23) for $\lambda = 0$;

3) an a priori estimate in W_0 uniform in λ for solutions $v(x,t)$ to (24), (23).

The continuity in λ is evident of the family $\{L_\varepsilon(\lambda)\}$ of operators. For $\lambda = 0$ and a fixed number ε, the boundary value problem (24), (23) is solvable in W_1 (see [28]) under the conditions of Theorem 1. Justify an a priori estimate in the space W_1 uniform in λ for all solutions $w(x,t)$ to (24), (23).

Let $w(x,t)$ be a solution to (24), (23) from W_1. Put $v(x,t) = w(x,t) + \delta(x,t,\lambda)w(0,t)$. The inequality

$$\int_0^T v^2(0,t)\,dt \le \int_0^T\int_0^1 v_x^2(x,t)\,dxdt + 2\int_0^T\int_0^1 v^2(x,t)\,dxdt, \tag{25}$$

easily implies that $v(x,t)$ belongs to W_1 too and is a solution to the boundary value problem

$$L_\varepsilon(\lambda)v \equiv v_{ttt} + v_{xx} + c(t)v - \varepsilon v_{xxttt} = f_2(x,t) + \lambda h_{1xx}(x,t)v(0,t), \tag{26}$$

$$v_x(0,t) = \lambda\alpha_1(t)v(0,t), \quad v_x(1,t) = \lambda\beta_1(t)v(0,t), \tag{27}$$

$$v(x,0) = v_t(x,0) = v(x,T) = 0. \tag{28}$$

Consider the equality

$$\int_0^T\int_0^1 L_\varepsilon(\lambda)v\cdot[-v+v_{ttt}](\lambda_0 - t)\,dxdt$$
$$= \int_0^T\int_0^1 [f_2(x,t) + \lambda h_{1xx}v(0,t)][-v+v_{ttt}](\lambda_0 - t)\,dxdt. \tag{29}$$

Integrating by parts and accounting for the above initial-boundary conditions (27), (28) for $v(x,t)$, we arrive at the equality

$$\int_0^T\int_0^1 \big[v_x^2 - c(t)v^2\big](\lambda_0 - t)\,dxdt + \frac{3}{2}\int_0^T\int_0^1 v_t^2\,dxdt + \frac{1}{2}\int_0^1 v_t^2(x,T)(\lambda_0 - T)\,dx$$
$$+\frac{3(1+\varepsilon)}{2}\int_0^T\int_0^1 v_{xt}^2\,dxdt + \frac{1+\varepsilon}{2}\int_0^1 v_{xt}^2(x,T)(\lambda_0 - T)\,dx$$
$$+\int_0^T\int_0^1 v_{ttt}^2(\lambda_0 - t)\,dxdt + \varepsilon\int_0^T\int_0^1 v_{xttt}^2(\lambda_0 - t)\,dxdt$$
$$+\varepsilon\lambda\int_0^T \big[\alpha_1(t)v_{ttt}^2(0,t) - \beta_1(t)v_{ttt}(0,t)v_{ttt}(1,t)\big](\lambda_0 - t)\,dt$$

$$= -\int_0^T\int_0^1 c(t)vv_{ttt}(\lambda_0-t)\,dxdt$$

$$+\varepsilon\lambda\int_0^T\{[3\beta_1'(t)v_{tt}(0,t)+3\beta_1''(t)v_t(0,t)+\beta_1'''(t)v(0,t)]v_{ttt}(1,t)$$

$$-[3\alpha_1'(t)v_{tt}(0,t)+3\alpha_1''(t)v_t(0,t)+\alpha_1'''(t)v(0,t)]v_{ttt}(0,t)\}(\lambda_0-t)\,dt$$

$$+\varepsilon\int_0^T\{[(v_{xtt}v_t)(1,t)-(v_{xtt}v_t)(0,t)](\lambda_0-t)+(v_{xt}v_t)(1,t)-(v_{xt}v_t)(0,t)\}\,dt$$

$$+\int_0^T\{[(v_xv+v_{xt}v_{tt})(1,t)-(v_xv+v_{xt}v_{tt})(0,t)](\lambda_0-t)+v_{xt}v_t(1,t)-v_{xt}v_t(0,t)\}\,dt$$

$$+\int_0^T\int_0^1[f_2(x,t)+\lambda h_{1xx}(x,t)v(0,t)](-v+v_{ttt})(\lambda_0-t)\,dxdt.$$

In view of (12), (13) for $\lambda_0=2T$, the Young inequality and (8)–(10) yield

$$\int_0^T\int_0^1\left[v_x^2-c(t)v^2\right]dxdt+\int_0^T\int_0^1 v_t^2\,dxdt+\int_0^1 v_t^2(x,T)\,dx$$

$$+(1+\varepsilon)\Big[\int_0^T\int_0^1 v_{xt}^2\,dxdt+\int_0^1 v_{xt}^2(x,T)\,dx\Big]+\int_0^T\int_0^1 v_{ttt}^2\,dxdt+\varepsilon\int_0^T\int_0^1 v_{xttt}^2\,dxdt$$

$$\le\frac{\varepsilon\lambda T}{4}\int_0^T v^2(1,t)\,dt+2T\delta_0\int_0^T\int_0^1(v^2+v_{ttt}^2)\,dxdt+\frac{T}{\delta_0}\int_0^T\int_0^1 f_2^2\,dxdt$$

$$+2Th_0\Big[\int_0^T v^2(0,t)\,dt+\frac12\int_0^T\int_0^1(v^2+v_{ttt}^2)\,dxdt\Big].$$

Hence,

$$\int_0^T\int_0^1\left[v_x^2-c(t)v^2\right]dxdt+\int_0^T\int_0^1 v_t^2\,dxdt\int_0^1 v_t^2(x,T)\,dx$$

$$+(1+\varepsilon)\Big[\int_0^T\int_0^1 v_{xt}^2\,dxdt+\int_0^1 v_{xt}^2(x,T)\,dx\Big]$$

$$+\int_0^T\int_0^1 v_{ttt}^2\,dxdt+\varepsilon\int_0^T\int_0^1 v_{xttt}^2\,dxdt$$

$$\le 2T\delta_0\int_0^T\int_0^1(v^2+v_{ttt}^2)\,dxdt+\frac{T}{\delta_0}\int_0^T\int_0^1 f_2^2\,dxdt$$

$$+2T\left(\frac{1}{8}+h_0\right)\left[\delta_1\int_0^T\int_0^1 v_x^2\,dxdt + C(\delta_1)\int_0^T\int_0^1 v^2\,dxdt\right]$$
$$+Th_0\int_0^T\int_0^1\left(v^2+v_{ttt}^2\right)dxdt, \tag{30}$$

where δ_1 is an arbitrary positive number satisfying the inequality

$$1-2T\left(\frac{1}{8}+h_0\right)\delta_1>0.$$

For a fixed δ_1 and $\delta_0=\frac{1}{4T}$, in view of the conditions of Theorem (11) there exists a sufficiently large $-c_0>0$ such that

$$-c_0-2T\delta_0-T\left(\frac{1}{4}+2h_0\right)C(\delta_1)-Th_0>0.$$

Therefore, (30) implies that

$$\int_0^T\int_0^1\left[v_x^2-c(t)v^2\right]dxdt+\int_0^T\int_0^1 v_t^2\,dxdt+\int_0^1 v_t^2(x,T)\,dx+$$
$$+(1+\varepsilon)\left[\int_0^T\int_0^1 v_{xt}^2\,dxdt+\int_0^1 v_{xt}^2(x,T)\,dx\right]+$$
$$+\int_0^T\int_0^1 v_{ttt}^2\,dxdt+\varepsilon\int_0^T\int_0^1 v_{xttt}^2\,dxdt\le M_1\int_0^T\int_0^1 f_2^2\,dxdt, \tag{31}$$

with a constant M_1 defined by c_0, $f(x,t)$, and $h(x,t)$.

Consider the equality

$$\int_0^T L_\varepsilon(\lambda)v\cdot\left[-v_{xxttt}+\left(x-\frac{1}{2}\right)v_{xttt}+v_{xx}+v_{ttt}\right]dxdt$$
$$=\int_0^T\int_0^1 F\cdot\left[-v_{xxttt}+\left(x-\frac{1}{2}\right)v_{xttt}+v_{xx}+v_{ttt}\right]dxdt,$$

$$F=f_2(x,t)+\lambda h_{1xx}(x,t)v(0,t).$$

Integrating by parts and (27), (28) for $v(x,t)$ ensures the equality

$$\left(1+\frac{3\varepsilon}{2}\right)\int_0^T\int_0^1 v_{xttt}^2\,dxdt+\int_0^T\int_0^1\left[v_{xx}^2+\frac{1}{2}v_{ttt}^2\right]dxdt+\int_0^1 v_{xt}^2(x,T)\,dx$$
$$+\varepsilon\int_0^T\int_0^1 v_{xxttt}^2\,dxdt+\frac{1+\varepsilon}{2}\int_0^1 v_{xxt}^2(x,T)\,dx$$

$$+\int_0^T \left\{ \left[\lambda(1+\varepsilon)\alpha_1(t)+\frac{1}{4}-\frac{\lambda\varepsilon}{4}(\alpha_1^2(t)+\beta_1^2(t))\right]v_{ttt}^2(0,t)\right.$$
$$\left. -\lambda(1+\varepsilon)\beta_1(t)v_{ttt}(0,t)v_{ttt}(1,t)+\frac{1}{4}v_{ttt}^2(1,t)\right\} dt$$
$$=\frac{\lambda\varepsilon}{4}\int_0^T [((\alpha_1'''(t))^2+(\beta_1'''(t))^2)v^2(0,t)+9((\alpha_1''(t))^2+(\beta_1''(t))^2)v_t^2(0,t)$$
$$+9((\alpha_1'(t))^2+(\beta_1'(t))^2)v_{tt}^2(0,t)+6(\alpha_1'''(t)\alpha_1''(t)+\beta_1'''(t)\beta_1''(t))v(0,t)v_t(0,t)$$
$$+6(\alpha_1'''(t)\alpha_1'(t)+\beta_1'''(t)\beta_1'(t))v(0,t)v_{tt}(0,t)+2(\alpha_1'''(t)\alpha_1(t)+\beta_1'''(t)\beta_1(t))v(0,t)v_{ttt}(0,t)$$
$$+18(\alpha_1''(t)\alpha_1'(t)+\beta_1''(t)\beta_1'(t))v_t(0,t)v_{tt}(0,t)+6(\alpha_1''(t)\alpha_1(t)+\beta_1''(t)\beta_1(t))v_t(0,t)v_{ttt}(0,t)$$
$$+6(\alpha_1'(t)\alpha_1(t)+\beta_1'(t)\beta_1(t))v_{tt}(0,t)v_{ttt}(0,t)]\,dt$$
$$+\int_0^T\int_0^1 \left[-c(t)v_xv_{xttt}-\left(x-\frac{1}{2}\right)v_{xx}v_{xttt}-\left(x-\frac{1}{2}\right)c(t)vv_{xttt}\right.$$
$$\left.-c(t)vv_{xx}-c(t)vv_{ttt}+F_xv_{xttt}+\left(x-\frac{1}{2}\right)Fv_{xttt}+Fv_{xx}+Fv_{ttt}\right] dxdt$$
$$-\lambda(1+\varepsilon)\int_0^T \{[3\alpha_1'(t)v_{tt}(0,t)+3\alpha_1''(t)v_t(0,t)+\alpha_1'''(t)v(0,t)]v_{ttt}(0,t)$$
$$-[3\beta_1'(t)v_{tt}(0,t)+3\beta_1''(t)v_t(0,t)+\beta_1'''(t)v(0,t)]v_{ttt}(1,t)\}dt$$
$$+\int_0^T [c(t)v(1,t)v_{xttt}(1,t)-c(t)v(0,t)v_{xttt}(0,t)-2v_{xt}(0,t)v_{tt}(0,t)$$
$$+2v_{xt}(1,t)v_{tt}(1,t)+F(0,t)v_{xttt}(0,t)-F(1,t)v_{xttt}(1,t)]\,dt.$$

Using (11)–(14), replacing $v_{xttt}(0,t)$ and $v_{xttt}(1,t)$ from (27) and estimating the summands on the right-hand side with the help of the Young inequality, (8) and (31), we obtain the a priori estimate

$$(1+\varepsilon)\int_0^T\int_0^1 v_{xttt}^2\,dxdt+\int_0^T\int_0^1 \left[v_{xx}^2+v_{ttt}^2\right] dxdt+\int_0^1 v_{xt}^2(x,T)\,dx$$
$$+\varepsilon\int_0^T\int_0^1 v_{xxttt}^2\,dxdt+(1+\varepsilon)\int_0^1 v_{xxt}^2(x,T)\,dx\le M_2\int_0^T\int_0^1 \left[f_2^2+f_{2x}^2\right] dxdt, \quad (32)$$

with the constant M_2 defined by c_0, $f(x,t)$, and $h(x,t)$.

Consider the equality

$$\int_0^T L_\varepsilon(\lambda)v_xv_{xxx}\,dxdt=\int_0^T\int_0^1 [f_{2x}(x,t)+\lambda h_{1xxx}(x,t)v(0,t)]v_{xxx}\,dxdt.$$

Integrating by parts and using the Young inequality, (31), (32), and the initial conditions (28) for $v(x,t)$, we arrive at the inequality

$$\int_0^T\int_0^1 v_{xxx}^2\,dxdt + \varepsilon\int_0^1 v_{xxxt}^2(x,T)\,dxdt \le M_3\int_0^T\int_0^1 f_{2x}^2\,dxdt \tag{33}$$

with constant M_3 defined by c_0, $f(x,t)$, and $hx,t)$.

Estimates (31)–(33) and equation (26) yield the estimate

$$\|v\|_{W_1} \le M_0 \tag{34}$$

uniform in λ.

These estimates allows us to apply the method of continuation in a parameter. Hence, under the conditions of the theorem, the boundary value problem (26)–(28) has a solution $v(x,t) = v_\varepsilon(x,t)$ from W_1 for all values of λ including $\lambda = 1$. Demonstrate that the family of solutions $\{v_\varepsilon(x,t)\}$ satisfies an a priori estimate uniform in ε which allows us to pass to the limit as $\varepsilon \to 0$.

The solutions $\{v_\varepsilon(x,t)\}$ to (26)–(28) satisfy (31)–(33). Choose a subsequence $\{\varepsilon_n\}$ such that $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to \infty$. By the theorem on weak closedness of a bounded set in $L_2(Q)$, there exist a sequence $\{v_m(x,t)\}$ and a function $v(x,t)$ such that

$$v_m(x,t) \to v(x,t) \text{ weakly in } W_{2,x,t}^{2,3}(Q),$$
$$v_{mx}(x,t) \to v_x(x,t) \text{ weakly in } W_{2,x,t}^{2,3}(Q),$$
$$\varepsilon_m w_{mxxttt}(x,t) \to 0 \text{ weakly in } L_2(Q)$$

as $m \to \infty$.

It is obvious that the limit function $v(x,t)$ satisfies (19). Put $w(x,t) = v(x,t) - \delta(x,t,1)v(0,t)$. The function $w(x,t)$ belongs to V_0 and it is a solution to (19)–(21).

It remains to show that a solution $u(x,t)$, $q(t)$ to the boundary value problem (1)–(4) is determined by $v(x,t)$. Indeed,

$$u_{xx}(x,t) = v(x,t), \tag{35}$$
$$u_x(0,t) = u_x(1,t) = 0. \tag{36}$$

The function $u(x,t)$ can be determined from these equalities. Let

$$w(x,t) = u_{ttt} + u_{xx} + c(t)u - f_1(x,t) - h_1(x,t)v(0,t). \tag{37}$$

In this case (16)–(18) imply that $w(x,t)$ satisfies the equalities

$$w_{xx}(x,t) = 0, \quad w_x(0,t) = w_x(1,t) = 0, \tag{38}$$

and so $w(x,t) \equiv 0$ for all $t \in [0,T]$.

Thus, $u(x,t)$ is a solution to the equation

$$u_{ttt} + u_{xx} + c(t)u - f_1(x,t) - h_1(x,t)v(0,t) = 0. \tag{39}$$

Taking $x = 0$ in (39), we conclude that

$$u_{ttt}(0,t) + u_{xx}(0,t) + c(t)u(0,t) - f_1(0,t) - h_1(0,t)v(0,t) = 0, \tag{40}$$

and so we arrive at the boundary value problem

$$u_{ttt}(0,t) + c(t)u(0,t) = 0, \tag{41}$$
$$u(0,0) = u_t(0,0) = u(0,T) = 0, \tag{42}$$

whose only solution is zero; i.e., $u(0,t) \equiv 0$.

In accord with the condition $u(0,t) = 0$ and (39), we derive that $u_{xx}(0,t) = v(0,t)$; i.e., $u(x,t)$ meets (2)–(4). Hence, our functions

$$u(x,t), \quad q(t) = \frac{u_{xx}(0,t) - f(0,t)}{h(0,t)}$$

belong to the required classes, satisfy (1) and define a solution to Inverse Boundary Value Problem 1. The theorem is proven.

4. Solvability of Boundary Value Problem 2

Introduce the notations

$$\triangle(t) = h_1(0,t)h_2(1,t) - h_2(0,t)h_1(1,t),$$

$$\tilde{f}_1(x,t) = \frac{h_1(x,t)}{\triangle(t)}(h_2(0,t)f(1,t) - h_2(1,t)f(0,t))$$

$$-\frac{h_2(x,t)}{\triangle(t)}(h_2(0,t)f(1,t) - h_2(1,t)f(0,t)) + f(x,t),$$

$$\alpha_0(t) = \tilde{f}_{1x}(0,t), \quad \beta_0(t) = \tilde{f}_{1x}(1,t),$$

$$\alpha_1(t) = \frac{1}{\triangle(t)}(h_2(1,t)h_{1x}(0,t) - h_1(1,t)h_{2x}(0,t)),$$

$$\alpha_2(t) = \frac{1}{\triangle(t)}(h_1(0,t)h_{2x}(0,t) - h_2(0,t)h_{1x}(0,t)),$$

$$\beta_1(t) = \frac{1}{\triangle(t)}(h_2(1,t)h_{1x}(1,t) - h_1(1,t)h_{2x}(1,t)),$$

$$\beta_2(t) = \frac{1}{\triangle(t)}(h_1(0,t)h_{2x}(1,t) - h_2(0,t)h_{1x}(1,t)),$$

$$\alpha(x,t) = \frac{1}{\triangle(t)}(h_2(1,t)h_{1xx}(x,t) - h_1(1,t)h_{2xx}(x,t)),$$

$$\beta(x,t) = \frac{1}{\triangle(t)}(h_1(0,t)h_{2xx}(x,t) - h_2(0,t)h_{1xx}(0,t).$$

We assume that

$$\begin{gathered} \tilde{f}_{1x}(0,t) + \tilde{f}_{1x}(1,t) = 0 \quad \text{for} \quad t \in [0,T], \\ \tilde{f}_{1x}(0,0) = \tilde{f}_{1xt}(0,0) = \tilde{f}_{1x}(0,T) = 0. \end{gathered} \tag{43}$$

Theorem 2. *Assume that*

$$\begin{gathered} c(t) \in C^1[0,T], \quad -c(t) \geq c_0 \gg 0 \quad \textit{for} \quad t \in [0,T], \\ h(x,t) \in C^3(\overline{Q}), \quad h_0 < \frac{1}{2T}, \quad \triangle(t) \neq 0, \end{gathered} \tag{44}$$

$$\begin{gathered} \alpha_1^2(t) + \beta_1^2(t) < \frac{1}{2}, \\ \frac{1}{2} + 2(\alpha_2(t)\beta_1(t) - \alpha_1(t)\beta_2(t)) \geq \alpha_1^2(t) + \beta_1^2(t) + \alpha_2^2(t) + \beta_2^2(t) \ \textit{for}\ t \in [0,T], \end{gathered} \tag{45}$$

$$\alpha_1(t)\xi_1^2 + [-\beta_1(t) + \alpha_2(t)]\,\xi_1\xi_2 - \beta_2(t)\xi_2^2 \geq 0 \quad \textit{for} \quad t \in [0,T], \quad (\xi_1,\xi_2) \in \mathbb{R}^2, \tag{46}$$

$$f(x,t) \in W_2^3(Q), \quad f_{xxttt}(x,t) \in L_2(Q). \tag{47}$$

Then there exists a regular solution to (5), (2), (3), (6) *such that* $u(x,t)$ *and* $u_{xx}(x,t)$ *belong to* $W^{2,3}_{2,x,t}(Q)$ *and* $q_1(t), q_2(t) \in L_2(0,T)$.

PROOF. Consider the auxiliary boundary value problem: *Find a solution* $u(x,t)$ *to*

$$u_{ttt} + u_{xx} + c(t)u = \tilde{f}_{1xx}(x,t) + \lambda[\alpha(x,t)u(0,t) + \beta(x,t)u(1,t)] \tag{48}$$

in Q *such that*

$$\begin{aligned} u_x(0,t) &= \alpha_1(t)u(0,t) + \alpha_2(t)u(1,t) + \alpha_0(t), \quad 0 < t < T, \\ u_x(1,t) &= \beta_1(t)u(0,t) + \beta_2(t)u(1,t) + \beta_0(t), \quad 0 < t < T, \end{aligned} \tag{49}$$

and

$$u(x,0) = u_t(x,0) = u(x,T) = 0, \quad x \in \Omega. \tag{50}$$

As in Section 3, instead of (48)–(50) we consider the boundary value problem: *Find a solution* $v(x,t)$ *to the equation*

$$v_{ttt} + v_{xx} + c(t)v = f_2(x,t) + \lambda[\alpha(x,t)v(0,t) + \beta(x,t)v(1,t)] \tag{51}$$

in Q *such that*

$$\begin{aligned} v_x(0,t) &= \alpha_1(t)v(0,t) + \alpha_2(t)v(1,t), \quad 0 < t < T, \\ v_x(1,t) &= \beta_1(t)v(0,t) + \beta_2(t)v(1,t), \quad 0 < t < T, \end{aligned} \tag{52}$$

$$v(x,0) = v_t(x,0) = v(x,T) = 0, \quad x \in \Omega. \tag{53}$$

where

$$f_2(x,t) = \tilde{f}_{1xx}(x,t) + B_0(x,t),$$

$$B_0(x,t) = \frac{1}{2}\lambda\beta(x,t)(\alpha_0(t) + \beta_0(t)) - \gamma_{ttt}(x,t) - \gamma_{xx}(x,t) - c(t)\gamma(x,t).$$

As in Section 3, without loss of generality we can consider the homogeneous initial conditions (53). Given $(x,t) \in \overline{Q}$ and $\lambda \in [0,1]$, we put

$$\gamma_1(x,t,\lambda) = \frac{\lambda x^2}{2}[\beta_1(t) - \alpha_1(t)] + \lambda x\alpha_1(t), \;\; \delta_1(x,t,\lambda) = \frac{\lambda x^2}{2}[\beta_2(t) - \alpha_2(t)] + \lambda x\alpha_2(t),$$
$$w(x,t) = v(x,t) - \gamma_1(x,t,\lambda)v(0,t) - \delta_1(x,t,\lambda)v(1,t),$$
$$\gamma_{11}(x,t,\lambda) = \gamma_1(x,t,\lambda) + \frac{\delta_1(x,t,\lambda)\gamma_1(1,t,\lambda)}{1 - \delta_1(1,t,\lambda)}, \quad \delta_{11}(x,t,\lambda) = \frac{\delta_1(x,t,\lambda)}{1 - \delta_1(1,t,\lambda)},$$
$$v(x,t) = w(x,t) + \gamma_{11}(x,t,\lambda)w(0,t) + \delta_{11}(x,t,\lambda)w(1,t).$$

Let $v(x,t)$ be a solution to (51). In this case $w(x,t)$ satisfies

$$w_{ttt} + w_{xx} + c(t)w = f_2(x,t) + \Phi(x,t,\lambda,\overline{w}(t)),$$

where

$$\overline{w}(t) = (w_{ttt}(0,t), w_{ttt}(1,t), w_{tt}(0,t), w_{tt}(1,t), w_t(0,t), w_t(1,t), w(0,t), w(1,t)),$$

$$\begin{aligned} \Phi(x,t,\lambda,\overline{w}(t)) = {} & A_1(x,t,\lambda)w_{ttt}(0,t) + A_2(x,t,\lambda)w_{ttt}(1,t) + A_3(x,t,\lambda)w_{tt}(0,t) \\ & + A_4(x,t,\lambda)w_{tt}(1,t) + A_5(x,t,\lambda)w_t(0,t) + A_6(x,t,\lambda)w_t(1,t) \\ & + A_7(x,t,\lambda)w(0,t) + A_8(x,t,\lambda)w(1,t), \end{aligned}$$

$$A_1(x,t,\lambda) = -\gamma_{11}(x,t,\lambda), \quad A_2(x,t,\lambda) = -\delta_{11}(x,t,\lambda),$$

$$A_3(x,t,\lambda) = -3\gamma_{11\,t}(x,t,\lambda), \quad A_4(x,t,\lambda) = -3\delta_{11\,t}(x,t,\lambda),$$

$$A_5(x,t,\lambda) = -3\gamma_{11\,tt}(x,t,\lambda), \quad A_6(x,t,\lambda) = -3\delta_{11\,tt}(x,t,\lambda),$$
$$A_7(x,t,\lambda) = \lambda\alpha - \gamma_{11\,ttt}(x,t,\lambda) - \gamma_{11\,xx}(x,t,\lambda) - c(t)\gamma_{11}(x,t,\lambda),$$
$$A_8(x,t,\lambda) = \lambda\beta - \delta_{11\,ttt}(x,t,\lambda) - \delta_{11\,xx}(x,t,\lambda) - c(t)\delta_{11}(x,t,\lambda).$$

Consider the auxiliary boundary value problem: *Find a solution $w(x,t)$ to*

$$w_{ttt} + w_{xx} + c(t)w = f_2(x,t) + \Phi(x,t,\lambda,\overline{w}(t)) \tag{54}$$

in Q such that

$$\begin{gathered} w_x(0,t) = w_x(1,t) = 0, \quad t \in (0,T), \\ w(x,0) = w_t(x,0) = w(x,T) = 0, \quad x \in \Omega. \end{gathered} \tag{55}$$

As in Section 3, this problem is solvable in V_0. To justify it, we need to utilize the methods of regularization and continuation in a parameter. The function $v(x,t)$ constructed allows us to define a solution $u(x,t)$, $q(t)$ to the boundary value problem (5), (2), (3), (6). The theorem is proven.

5. Conclusions

1. Conditions (12) are some smallness conditions for Inverse Boundary Value Problem 1. Obviously, the set of data $f(x,t)$ and $h(x,t)$ satisfying this condition is not empty. Similarly, the smallness conditions (44) and (45) for Inverse Problem 2 also make some sense.

2. Note that the fulfilment of (45) implies the nonnegative definiteness of the quadratic form

$$\begin{gathered} \left[\frac{1}{2} - \alpha_1^2(t) - \beta_1^2(t)\right]\xi_1^2 - 2\left[\alpha_1(t)\alpha_2(t) + \beta_1(t)\beta_2(t)\right]\xi_1\xi_2 \\ + \left[\frac{1}{2} - \alpha_2^2(t) - \beta_2^2(t)\right]\xi_2^2 \geq 0 \end{gathered}$$

for $t \in [0,T]$ and $(\xi_1,\xi_2) \in \mathbb{R}^2$.

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N. N. Nikolaev
Nyurba Technical Lyceum, Nyurba, Russia
nuke1987@mail.ru

S. V. Popov
North-Eastern Federal University
Institute of Mathematics and Informatics, Yakutsk, Russia
madu@ysu.ru

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COEFFICIENT INVERSE PROBLEMS FOR HIGHER ORDER QUASIHYPERBOLIC EQUATIONS WITH INTEGRAL OVERDETERMINATION

S. S. Pavlov

Abstract. We study a linear coefficient inverse problem for higher order quasihyperbolic equations. Solvability is established of an inverse problem of determining a solution and an external force for higher order equations in time with an integral overdetermination condition. Existence and uniqueness theorems are proven for solutions to these coefficient inverse problems.

Keywords: inverse problem, integral ovedetermination condition, method of continuation in a parameter, a priori estimate, existence, uniqueness

Introduction

We study the linear coefficient inverse problem for quasihyperbolic equations which is reduced to finding functions $u(x,t)$ and $q(t)$ in the equation

$$(-1)^{m-1}D_t^{2m}u - \Delta u + c(x,t)u = f(x,t) + q(t)h(x,t) \qquad \left(D_t^k = \frac{\partial^k}{\partial t^k}\right).$$

For the first time, some well-posed boundary value problem for quasihyperbolic equations was stated by V. N. Vragov in [1]. A series of results on solvability of boundary value problems and properties of solutions to these equations are exposed in the articles by I. E. Egorov V. E. Fedorov [2, 3], A. N. Terekhov [4], A. I. Kozhanov and E. F. Sharin [5]. Inverse problems for quasihyperbolic equations were not considered by now.

Linear and nonlinear inverse problems for hyperbolic equations in various settings are studied in the articles by M. M. Lavrent′ev [6], V. G. Romanov [7], Yu. E. Anikonov [8–11], Yu. Ya. Belov [12, 13], B. A. Bubnov [14, 15], S. I. Kabanikhin [16], A. I. Prilepko [17], etc. The problems of determining coefficients of hyperbolic equations on some additional information about a solution are of great importance for practice [18–20].

The articles [21, 22] are devoted to the study of an inverse problem of recovering an external force for a hyperbolic equation with an integral overdetermination condition. Similar inverse problems for hyperbolic equations were studied earlier in [23–27]. Solvability of an inverse problem for a hyperbolic equation with several unknown sources is studied in the articles by I. R. Valitov and A. I. Kozhanov [25].

1. Statement of the Problem

Let Ω be a bounded domain in $\mathbb{R}$ with boundary Γ, $Q = \Omega \times (0,T)$ is a cylinder with the lateral boundary $S = \Gamma \times (0,T)$, $f(x,t)$, $c(x,t)$, $h(x,t)$, and $K(x,t)$ are given functions, and m is a positive integer.

Inverse problem I. *Find $u(x,t)$ and $q(t)$ satisfying the equation*

$$(-1)^{m-1}D_t^{2m}u - \Delta u + c(x,t)u = f(x,t) + q(t)h(x,t), \tag{1}$$

in Q, the boundary conditions

$$\begin{aligned} D_t^i u(x,0) = 0, \quad i = \overline{0,m}, \\ D_t^j u(x,T) = 0, \quad j = \overline{1,m-1}, \quad x \in \Omega, \end{aligned} \tag{2}$$

$$u(x,t)|_S = 0, \tag{3}$$

and the overdetermination conditions

$$\int\limits_\Omega K(x,t)u(x,t)\,dx = 0 \quad \text{for} \quad t \in (0,T). \tag{4}$$

Inverse Problem II. *Find $u(x,t)$ and $q(t)$ satisfying* (1) *in Q, the boundary conditions* (2), *the ovedetermination conditions* (4), *and such that*

$$\left.\frac{\partial u(x,t)}{\partial \nu}\right|_S = 0, \tag{5}$$

(ν *is the inner normal to* Γ).

Without loss of generality, we can consider the case of $m = 2$. In the case of $m > 2$, all arguments are quite similar to this case but more cumbersome.

Introduce the notations

$$h_0(t) = \int\limits_\Omega K(x,t)h(x,t)\,dx, \quad \alpha_0(t) = \frac{1}{h_0(t)}\int\limits_\Omega K(x,t)f(x,t)\,dx,$$

$$f_1(x,t) = f(x,t) - \alpha_0(t)h(x,t), \quad H_1 = \max_{\overline{Q}} |h(x,t)|,$$

$$c_1 = \max_{i=1,\dots,n}\max_{\overline{Q}} |c_{x_i}(x,t)|, \quad H_2 = \max_{i=1,\dots,n}\max_{\overline{Q}} |h_{x_i}(x,t)|,$$

$$M_1 = \max_{t\in[0,T]}\left(\frac{3}{h_0^2(t)}\int\limits_Q\left(\int\limits_\Omega (K^2(y,t)\,dy\right)dxdt\right),$$

$$M_2 = \max_{t\in[0,T]}\left(\frac{3}{h_0^2(t)}\int\limits_Q\left(\int\limits_\Omega (K(y,t)c(y,t))^2\,dy\right)dxdt\right),$$

$$M_3 = \max_{t\in[0,T]}\left(\frac{3}{h_0^2(t)}\int\limits_Q\left(\int\limits_\Omega K^2(y,t)\,dy\right)dxdt\right),$$

$$M_0 = \max\{M_1, M_2, M_3\}.$$

Denote by V the anisotropic space of functions having generalized derivatives in space and time variables up to the second order and forth order, respectively. The space V is endowed with the natural norm

$$\|u\|_V = \left(\int\limits_Q\left[u^2 + \sum_{i,j=1}^n u_{x_ix_j}^2 + \left(D_t^4 u\right)^2\right]dxdt\right)^{\frac{1}{2}}.$$

Theorem 1. *Assume that*

$$h(x,t) \in C^1(\overline{Q}), \quad c(x,t) \in C^4(\overline{Q}), \quad K(x,t) \in C^4(\overline{Q}),$$

$$f(x,t), f_{x_i}(x,t) \in L_2(Q), \quad i = 1, \dots, n,$$

$$h_0(t) \neq 0,\ t \in [0,T], \quad c(x,T) \geq\ 0 \text{ for } x \in \overline{\Omega},$$

$$f(x,t)|_S = h(x,t)|_S = 0$$

and there exists λ_0*:* $\lambda_0 > T$, $[(\lambda_0 - t)c(x,t)]_t \leq 0$ *for* $(x,t) \in \overline{Q}$,

$$M_0 \left(\lambda_0^2 T^2 \left(H_1^2 + \frac{H_2^2}{2(1 - \lambda_0^2 c_1^2 T^6)} \right) + 2H_1^2 \right) < 1.$$

Then there exist $u(x,t) \in V$ *and* $q(t) \in L_2(0,T)$ *that solve the inverse problem* (1)–(4).

Proof. We start with some auxiliary constructions. Multiply (1) by $K(x,t)$ and integrate the result over Ω. We can calculate $q(t)$ from the equality obtained as follows:

$$q(t) = Z(t,u) - \alpha_0(t),$$

$$Z(t,u) = \frac{1}{h_0(t)} \Bigg[- \int\limits_{\Omega} K(x,t) u_{tttt}(x,t)\, dx$$

$$- \int\limits_{\Omega} K(x,t) \Delta u(x,t)\, dx + \int\limits_{\Omega} K(x,t) c(x,t) u(x,t)\, dx \Bigg].$$

Let us consider

Auxiliary Boundary Value Problem: *Find a solution* $u(x,t)$ *to the equation*

$$-u_{tttt} - \Delta u + c(x,t)u = f_1(x,t) + Z(t,u)h(x,t), \tag{6}$$

in Q *satisfying*

$$u(x,0) = u_t(x,0) = u_{tt}(x,0) = 0, \quad u_t(x,T) = 0, \quad x \in \Omega, \tag{7}$$

$$u|_S = 0. \tag{8}$$

We will prove that this problem is solvable in V. To this end, we employ regularization and the method of continuation in a parameter.

Let ε be a positive real number. Consider the new boundary value problem of finding a solution $u(x,t)$ to the equation

$$-u_{tttt} - \Delta u + c(x,t)u - \varepsilon \Delta u_t = f_1(x,t) + Z(t,u)h(x,t) \tag{9}$$

in Q satisfying (7) and (8).

Apply the method of continuation in a parameter.

Let λ be a number in $[0,1]$. Examine the following problem: Find a solution $u(x,t)$ to

$$-u_{tttt} - \Delta u + c(x,t)u - \varepsilon \Delta u_t = f_1(x,t) + \lambda Z(t,u)h(x,t) \tag{10}$$

in Q satisfying (7) and (8).

Demonstrate that if ε is a fixed number and $f_1(x,t)$ belongs to the $L_2(Q)$ space then the boundary value problem $(10_{\varepsilon,\lambda})$, (7), (8) is solvable in V.

In accord with the theorem on the method of continuation in a parameter [28, Chapter 3, Section 14], the problem $(10_{\varepsilon,\lambda})$, (7), (8) is solvable in the space V

whenever it is solvable for $\lambda = 0$ in V and all solutions satisfy an a priori estimate uniform (in $\lambda \in [0,1]$) in the same space [3, 29].

Solvability of the boundary value problem ($10_{\varepsilon,0}$), (7), (8) for a fixed ε and $f_1(x,t)$ from $L_2(Q)$ is known [3]. Show that all solutions to the boundary value problem ($10_{\varepsilon,\lambda}$), (7), (8) from V satisfy the required a priori estimate.

Consider the equality

$$\int\limits_Q [-u_{tttt}(x,t) - \Delta u(x,t) + c(x,t)u(x,t) - \varepsilon\Delta u_t(x,t)]u_t(x,t)(\lambda_0 - t)\,dxdt$$
$$= \int\limits_Q [f_1(x,t) + \lambda Z(t,u)h(x,t)]u_t(x,t)(\lambda_0 - t)\,dxdt, \tag{11}$$

which is a consequence of ($10_{\varepsilon,\lambda}$). We have

$$-\int\limits_\Omega\int\limits_0^T u_{tttt}(x,t)u_t(x,t)(\lambda_0 - t)\,dxdt$$
$$= \frac{1}{2}\int\limits_\Omega\int\limits_0^T \frac{\partial(u_{tt}^2(x,t)(\lambda_0 - t))}{\partial t}\,dxdt + \frac{3}{2}\int\limits_\Omega\int\limits_0^T u_{tt}^2(x,t)\,dxdt$$
$$= \frac{(\lambda_0 - T)}{2}\int\limits_\Omega u_{tt}^2(x,T)\,dx + \frac{3}{2}\int\limits_\Omega\int\limits_0^T u_{tt}^2(x,t)\,dxdt,$$

$$-\int\limits_\Omega\int\limits_0^T \Delta u(x,t)u_t(x,t)(\lambda_0 - t)\,dxdt$$
$$= \frac{1}{2}\sum_{i=1}^n\int\limits_\Omega\int\limits_0^T \frac{\partial(u_{x_i}^2(x,t)(\lambda_0 - t))}{\partial t}\,dxdt + \frac{1}{2}\sum_{i=1}^n\int\limits_\Omega\int\limits_0^T u_{x_i}^2(x,t)\,dxdt$$
$$= \frac{(\lambda_0 - T)}{2}\sum_{i=1}^n\int\limits_\Omega u_{x_i}^2(x,T)\,dx + \frac{1}{2}\sum_{i=1}^n\int\limits_\Omega\int\limits_0^T u_{x_i}^2(x,t)\,dxdt,$$

$$\int\limits_\Omega\int\limits_0^T c(x,t)u(x,t)u_t(x,t)(\lambda_0 - t)\,dxdt = \frac{1}{2}\int\limits_\Omega\int\limits_0^T \frac{\partial(u^2(x,t)c(x,t)(\lambda_0 - t))}{\partial t}\,dxdt$$
$$+\frac{1}{2}\int\limits_\Omega\int\limits_0^T (c(x,t) - c_t(x,t)(\lambda_0 - t))u^2(x,t)\,dxdt = \frac{(\lambda_0 - T)}{2}\int\limits_\Omega u^2(x,T)c(x,T)\,dx$$
$$+\frac{1}{2}\int\limits_\Omega\int\limits_0^T (c(x,t) - c_t(x,t)(\lambda_0 - t))u^2(x,t)\,dxdt,$$

$$-\varepsilon\int\limits_\Omega\int\limits_0^T \Delta u_t(x,t)u_t(x,t)(\lambda_0 - t)\,dxdt = \varepsilon\sum_{i=1}^n\int\limits_\Omega\int\limits_0^T u_{x_it}^2(x,t)(\lambda_0 - t)\,dxdt.$$

Integrating by parts and using the Young inequality and (11), we can easily derive that

$$\begin{gathered}\frac{3}{2}\int\limits_Q u_{tt}^2(x,t)\,dxdt + \frac{1}{2}\sum_{i=1}^n \int\limits_Q u_{x_i}^2(x,t)\,dxdt \\ +\frac{1}{2}\int\limits_Q (c(x,t)-c_t(x,t)(\lambda_0-t))u^2(x,t)\,dxdt \\ +\varepsilon\sum_{i=1}^n\int\limits_Q u_{x_it}^2(x,t)(\lambda_0-t)\,dxdt + \frac{(\lambda_0-T)}{2}\int\limits_\Omega u_{tt}^2(x,T)\,dx \\ +\frac{(\lambda_0-T)}{2}\sum_{i=1}^n\int\limits_\Omega u_{x_i}^2(x,T)\,dx + \frac{(\lambda_0-T)}{2}\int\limits_\Omega c(x,T)u^2(x,T)\,dx \\ \le \frac{\delta_1^2+\delta_2^2}{2}\int\limits_Q u_t^2(x,t)\,dxdt + \frac{1}{2\delta_1^2}\int\limits_Q f_1^2(x,t)(\lambda_0-t)^2\,dxdt \\ +\frac{1}{2\delta_2^2}\int\limits_Q (\lambda_0-t)^2 Z^2(t,u)h^2(x,t)\,dxdt\end{gathered}$$

(here δ_1 and δ_2 are arbitrary positive numbers).

The inequality

$$\int\limits_Q u_t^2(x,t)\,dxdt \le T^2\int\limits_Q u_{tt}^2(x,t)\,dxdt \tag{12}$$

holds. Let $\delta_1=\delta_2=\frac{1}{T}$. In view of (12) and the conditions of Theorem 1, we infer

$$\begin{gathered}\int\limits_Q u_{tt}^2(x,t)\,dxdt + \sum_{i=1}^n\int\limits_Q u_{x_i}^2(x,t)\,dxdt \\ +\int\limits_Q (c(x,t)-c_t(x,t)(\lambda_0-t))u^2(x,t)\,dxdt \\ +2\varepsilon\lambda_0\sum_{i=1}^n\int\limits_Q u_{x_it}^2(x,t)\,dxdt + (\lambda_0-T)\int\limits_\Omega u_{tt}^2(x,T)\,dx \\ +(\lambda_0-T)\sum_{i=1}^n\int\limits_\Omega u_{x_i}^2(x,T)\,dx \\ \le \lambda_0^2T^2\int\limits_Q f_1^2(x,t)\,dxdt + \lambda_0^2\,H_1^2T^2\int\limits_Q Z^2(t,u)\,dxdt.\end{gathered} \tag{13}$$

Multiply (10) by $(-\Delta u_t(x,t))(\lambda_0-t)$. Integrating by parts on the left-hand side and applying the Young inequality on the right-hand side, we justify that

$$
\begin{aligned}
&3\sum_{i=1}^{n}\int\limits_{Q} u_{x_itt}^2(x,t)\,dxdt+\int\limits_{Q}[\Delta u(x,t)]^2\,dxdt\\
&+\sum_{i=1}^{n}\int\limits_{Q} u_{x_i}^2(x,t)[c(x,t)-c_t(x,t)(\lambda_0-t)]\,dxdt\\
&+(\lambda_0-T)\sum_{i=1}^{n}\int\limits_{\Omega} u_{x_itt}^2(x,T)\,dx+(\lambda_0-T)\sum_{i=1}^{n}\int\limits_{\Omega} c(x,T)u_{x_i}^2(x,T)\,dx\\
&+2\varepsilon\int\limits_{Q}[\Delta u_t(x,t)]^2(\lambda_0-t)\,dxdt\\
&\le \delta_3^2\int\limits_{Q}[\Delta u_t(x,t)]^2(\lambda_0-t)\,dxdt+\frac{1}{\delta_3^2}\int\limits_{Q} f_1^2(x,t)(\lambda_0-t)\,dxdt\\
&+\delta_4^2\sum_{i=1}^{n}\int\limits_{Q} u_{x_it}^2\,dxdt+\frac{1}{\delta_4^2}\sum_{i=1}^{n}\int\limits_{Q} h_{x_i}^2(x,t)(\lambda_0-t)^2Z^2(t,u)\,dxdt\\
&+\delta_5^2\sum_{i=1}^{n}\int\limits_{Q}(\lambda_0-t)^2c_{x_i}^2(x,t)u^2(x,t)\,dxdt+\frac{1}{\delta_5^2}\sum_{i=1}^{n}\int\limits_{Q} u_{x_it}^2(x,t)\,dxdt.
\end{aligned}
$$

Put

$$
\delta_3=\sqrt{\varepsilon},\quad \delta_4=\frac{\sqrt{2(1-\lambda_0^2\,c_1^2\,T^6)}}{T},\quad \delta_5=\frac{\sqrt{2}}{2\lambda_0c_1T^2}.
$$

Taking (12) and the conditions of Theorem 1 into account, we find that

$$
\begin{aligned}
&\sum_{i=1}^{n}\int\limits_{Q} u_{x_itt}^2(x,t)\,dxdt+\int\limits_{Q}[\Delta u(x,t)]^2\,dxdt\\
&+\sum_{i=1}^{n}\int\limits_{Q}(c(x,t)-c_t(x,t)(\lambda_0-t))u_{x_i}^2(x,t)\,dxdt\\
&+\varepsilon\int\limits_{Q}[\Delta u_t(x,t)]^2(\lambda_0-t)\,dxdt+(\lambda_0-T)\sum_{i=1}^{n}\int\limits_{\Omega} u_{x_itt}^2(x,T)\,dx\\
&\le \frac{1}{2}\int\limits_{Q} u_{tt}^2(x,t)\,dxdt+\frac{1}{\varepsilon}\int\limits_{Q} f_1^2(x,t)(\lambda_0-t)\,dxdt\\
&+\frac{\lambda_0^2\,H_2^2\,T^2}{2(1-\lambda_0^2c_1^2T^6)}\int\limits_{Q} Z^2(t,u)\,dxdt. \qquad (14)
\end{aligned}
$$

Consider the inequality

$$
\begin{aligned}
&\int\limits_{Q}(-u_{tttt}(x,t)-\Delta u(x,t)+c(x,t)u(x,t)-\varepsilon\Delta u_t(x,t))(-u_{tttt}(x,t))\,dxdt\\
&=\int\limits_{Q}\left(f_1(x,t)+\lambda Z(t,u)h(x,t)\right)(-u_{tttt}(x,t))\,dxdt. \qquad (15)
\end{aligned}
$$

Put $\delta_7^2 = \delta_8^2 = \frac{1}{2}$. Integrating by parts and involving the Young inequality and (15), we infer

$$\int\limits_Q u_{tttt}^2(x,t)\,dxdt - 2\sum_{i=1}^{n}\int\limits_Q u_{x_itt}^2(x,t)\,dxdt - 2\int\limits_Q c(x,t)u_{tt}^2(x,t)\,dxdt$$
$$-\int\limits_Q u^2(x,t)c_{tttt}(x,t)\,dxdt + 4\int\limits_Q u_t^2(x,t)c_{tt}(x,t)\,dxdt$$
$$+\varepsilon\sum_{i=1}^{n}\int\limits_\Omega u_{x_itt}^2(x,T)\,dx + \int\limits_\Omega u^2(x,T)c_{ttt}(x,T)\,dx$$
$$= -2\int\limits_\Omega c_t(x,T)u(x,T)u_{tt}(x,T)\,dx - 2\int\limits_\Omega c(x,T)u(x,T)u_{ttt}(x,T)\,dx$$
$$\leq 2\int\limits_Q f_1^2(x,t)\,dxdt + 2H_1^2\int\limits_Q Z^2(t,u)\,dxdt. \tag{16}$$

Inequalities (13), (14), and (16) yield

$$\int\limits_Q u_{tttt}^2(x,t)\,dxdt + \int\limits_Q [\Delta u(x,t)]^2\,dxdt + \varepsilon\lambda_0\int\limits_Q [\Delta u_t(x,t)]^2\,dxdt$$
$$++\sum_{i=1}^{n}\int\limits_Q [1+(c(x,t)-c_t(x,t)(\lambda_0-t))]u_{x_i}^2(x,t)\,dxdt$$
$$+\int\limits_Q \left(\frac{1}{2}-2c(x,t)\right)u_{tt}^2(x,t)\,dxdt + \int\limits_Q (c(x,t)-c_t(x,t)(\lambda_0-t)$$
$$-c_{tttt}(x,t))u^2(x,t)\,dxdt - \sum_{i=1}^{n}\int\limits_Q u_{x_itt}^2(x,t)\,dxdt$$
$$+2\varepsilon\lambda_0\sum_{i=1}^{n}\int\limits_Q u_{x_it}^2(x,t)\,dxdt + (\lambda_0-T)\sum_{i=1}^{n}\int\limits_\Omega u_{x_i}^2(x,T)\,dx$$
$$+4\int\limits_Q c_{tt}(x,t)u_t^2(x,t)\,dxdt + (\varepsilon+(\lambda_0-T))\sum_{i=1}^{n}\int\limits_\Omega u_{x_itt}^2(x,T)\,dx$$
$$+(\lambda_0-T)\int\limits_\Omega u_{tt}^2(x,T)\,dx \leq \left(\lambda_0^2T^2+\frac{\lambda_0}{\varepsilon}+2\right)\int\limits_Q f_1^2(x,t)\,dxdt$$
$$+\left(\lambda_0^2T^2\left(H_1^2+\frac{H_2^2}{2(1-\lambda_0^2c_1^2T^6)}\right)+2H_1^2\right)\int\limits_Q Z^2(t,u)\,dxdt. \tag{17}$$

The inequality

$$(a_1+\cdots+a_p)^2 \leq p\big(a_1^2+\cdots+a_p^2\big),$$

and the Hölder inequality imply that

$$Z^2(t,u) \le \frac{3}{h_0^2(t)}\Bigg[\Bigg(\int\limits_\Omega K(y,t)u_{tttt}(y,t)\,dy\Bigg)^2 + \Bigg(\int\limits_\Omega K(y,t)\Delta u(y,t)\,dy\Bigg)^2 + \Bigg(\int\limits_\Omega K(y,t)c(x,t)u(y,t)\,dy\Bigg)^2\Bigg].$$

The inequalities

$$\begin{aligned}
&\frac{3}{h_0^2(t)}\Bigg(\int\limits_Q K(y,t)u_{tttt}(y,t)\,dy\Bigg)^2\\
&\le \frac{3}{h_0^2(t)}\int\limits_Q\Bigg(\int\limits_\Omega K^2(y,t)\,dy\Bigg)\Bigg(\int\limits_\Omega u_{tttt}^2(y,t)\,dy\Bigg)\,dxdt\\
&\le \frac{3}{h_0^2(t)}\int\limits_Q\Bigg(\int\limits_\Omega (K^2(y,t)\,dy\Bigg)\,dxdt\int\limits_Q\Bigg(\int\limits_\Omega u_{tttt}^2(y,t)\,dy\Bigg)\,dxdt\\
&\le \max_T\Bigg(\frac{3}{h_0^2(t)}\int\limits_Q\Bigg(\int\limits_\Omega (K^2(y,t)\,dy\Bigg)\,dxdt\Bigg)\int\limits_Q u_{tttt}^2(x,t)\,dxdt\\
&\le M_1\int\limits_Q u_{tttt}^2(x,t)\,dxdt,
\end{aligned}$$

$$\begin{aligned}
&\frac{3}{h_0^2(t)}\Bigg(\int\limits_Q K(y,t)c(y,t)u(y,t)\,dy\Bigg)^2\\
&\le \frac{3}{h_0^2(t)}\int\limits_Q\Bigg(\int\limits_\Omega (K(y,t)c(y,t))^2\,dy\Bigg)\Bigg(\int\limits_\Omega u^2(y,t)\,dy\Bigg)\,dxdt\\
&\le \frac{3}{h_0^2(t)}\int\limits_Q\Bigg(\int\limits_\Omega (K(y,t)c(y,t))^2\,dy\Bigg)\,dxdt\int\limits_Q\Bigg(\int\limits_\Omega u^2(y,t)\,dy\Bigg)\,dxdt\\
&\le \max_T\Bigg(\frac{3}{h_0^2(t)}\int\limits_Q\Bigg(\int\limits_\Omega (K(y,t)c(y,t))^2\,dy\Bigg)\,dxdt\Bigg)\int\limits_Q u^2(x,t)\,dxdt\\
&\le M_2\int\limits_Q u^2(x,t)\,dxdt,
\end{aligned}$$

$$\begin{aligned}
&\frac{3}{h_0^2(t)}\Bigg(\int\limits_Q K(y,t)\Delta u(y,t)\,dy\Bigg)^2\\
&\le \frac{3}{h_0^2(t)}\int\limits_Q\Bigg(\int\limits_\Omega K^2(y,t)\,dy\Bigg)\Bigg(\int\limits_\Omega [\Delta u(y,t)]^2\,dy\Bigg)\,dxdt\\
&\le \frac{3}{h_0^2(t)}\int\limits_Q\Bigg(\int\limits_\Omega K^2(y,t)\,dy\Bigg)\,dxdt\int\limits_Q\Bigg(\int\limits_\Omega [\Delta u(y,t)]^2\,dy\Bigg)\,dxdt
\end{aligned}$$

$$\leq \max_{T}\left(\frac{3}{h_0^2(t)}\int\limits_{Q}\left(\int\limits_{\Omega} K^2(y,t)\,dy\right)dxdt\right)\int\limits_{Q}[\Delta u(x,t)]^2\,dxdt$$
$$\leq M_3\int\limits_{Q}[\Delta u(x,t)]^2\,dxdt,$$

hold, where the constants M_1, M_2, and M_3 are defined by $h(x,t)$, $K(x,t)$, and the domain Ω,

$$\left(\lambda_0^2T^2\left(H_1^2+\frac{H_2^2}{2(1-\lambda_0^2c_1^2T^6)}\right)+2H_1^2\right)\int\limits_{Q} Z^2(t,u)\,dxdt$$
$$\leq M_0\left(\lambda_0^2T^2\left(H_1^2+\frac{H_2^2}{2(1-\lambda_0^2c_1^2T^6)}\right)+2H_1^2\right)$$
$$\times\int\limits_{Q}\left(\int\limits_{\Omega}u^2(y,t)\,dy+\int\limits_{\Omega}u_{tttt}^2(y,t)\,dy+\int\limits_{\Omega}[\Delta u(y,t)]^2\,dy\right)dxdt$$
$$\leq M_0\left(\lambda_0^2T^2\left(H_1^2+\frac{H_2^2}{2(1-\lambda_0^2c_1^2T^6)}\right)+2H_1^2\right)$$
$$\times\int\limits_{Q}\left(u^2(x,t)+u_{tttt}^2(x,t)+[\Delta u(x,t)]^2\right)dxdt. \tag{18}$$

Inequalities (18) and (17) and the conditions of Theorem 1 validate the a priori estimate

$$\left(1-M_0\left(\lambda_0^2T^2\left(H_1^2+\frac{H_2^2}{2(1-\lambda_0^2c_1^2T^6)}\right)+2H_1^2\right)\right)$$
$$\times\left(\int\limits_{Q}u_{tttt}^2(x,t)\,dxdt+\int\limits_{Q}[\Delta u(x,t)]^2\,dxdt\right)$$
$$+\varepsilon\lambda_0\int\limits_{Q}[\Delta u_t(x,t)]^2\,dxdt+\sum_{i=1}^{n}\int\limits_{Q}[1+(c(x,t)-c_t(x,t)(\lambda_0-t))]u_{x_i}^2(x,t)\,dxdt$$
$$+\int\limits_{Q}\left(\frac{1}{2}-2c(x,t)\right)u_{tt}^2(x,t)\,dxdt+\int\limits_{Q}\Bigg(c(x,t)-c_t(x,t)(\lambda_0-t)-c_{tttt}(x,t)$$
$$-M_0\left(\lambda_0^2T^2\left(H_1^2+\frac{H_2^2}{2(1-\lambda_0^2c_1^2T^6)}\right)+2H_1^2\right)\Bigg)u^2(x,t)\,dxdt$$
$$-\sum_{i=1}^{n}\int\limits_{Q}u_{x_itt}^2(x,t)\,dxdt+2\varepsilon\lambda_0\sum_{i=1}^{n}\int\limits_{Q}u_{x_it}^2(x,t)\,dxdt$$
$$+(\lambda_0-T)\sum_{i=1}^{n}\int\limits_{\Omega}u_{x_i}^2(x,T)\,dx+4\int\limits_{Q}c_{tt}(x,t)u_t^2(x,t)\,dxdt$$
$$+(\varepsilon+(\lambda_0-T))\sum_{i=1}^{n}\int\limits_{\Omega}u_{x_itt}^2(x,T)\,dx+(\lambda_0-T)\int\limits_{\Omega}u_{tt}^2(x,T)\,dx$$
$$\leq\left(\lambda_0^2T^2+\frac{\lambda_0}{\varepsilon}+2\right)\int\limits_{Q}f_1^2(x,t)\,dxdt+M_4, \tag{19}$$

where the constant M_4 is defined by $h(x,t)$, $f(x,t)$, and $K(x,t)$ and ε, T, λ_0, and H_1.

Estimate (19) and the theorem on the method of continuation in a parameter [28, Chapter 3, Section 14] imply that for a fixed ε the boundary value problem $(10_{\varepsilon,\lambda})$, (7), (8) is solvable in V for all λ in $[0,1]$. In other words the boundary value problem $(10_{\varepsilon,\lambda})$, (7), (8) has a solution $u_\varepsilon(x,t)$, from V.

Next, we obtain a priori estimates uniform in ε and justify the passage to the limit as $\varepsilon \to 0$.

By the same arguments as those for obtaining (19), but integrating over x_i in the summands with $f_1(x,t)$, we demonstrate that the family of functions $\{u_\varepsilon(x,t)\}$ satisfies the inequality

$$\int\limits_Q u_{tttt}^2(x,t)\,dxdt + \int\limits_Q [\Delta u(x,t)]^2\,dxdt + \varepsilon\lambda_0 \int\limits_Q [\Delta u_t(x,t)]^2\,dxdt$$
$$+\sum_{i=1}^n \int\limits_Q [1 + (c(x,t) - c_t(x,t)(\lambda_0 - t))]u_{x_i}^2(x,t)\,dxdt$$
$$+\int\limits_Q \left(\frac{1}{2} - 2c(x,t)\right) u_{tt}^2(x,t)\,dxdt$$
$$+\int\limits_Q (c(x,t) - c_t(x,t)(\lambda_0 - t) - c_{tttt}(x,t))u^2(x,t)\,dxdt$$
$$-\sum_{i=1}^n \int\limits_Q u_{x_itt}^2(x,t)\,dxdt + 2\varepsilon\lambda_0 \sum_{i=1}^n \int\limits_Q u_{x_it}^2(x,t)\,dxdt$$
$$+(\lambda_0 - T)\sum_{i=1}^n \int\limits_\Omega u_{x_i}^2(x,T)\,dx + 4\int\limits_Q c_{tt}(x,t)u_t^2(x,t)\,dxdt$$
$$+(\varepsilon + (\lambda_0 - T))\sum_{i=1}^n \int\limits_\Omega u_{x_itt}^2(x,T)\,dx + (\lambda_0 - T)\int\limits_\Omega u_{tt}^2(x,T)\,dx$$
$$\le \left(2 + \lambda_0^2T^2\right)\int\limits_Q f_1^2(x,t)\,dxdt + \frac{1}{2(H_2^2\lambda_0^2 + 2\lambda_0^2c_1^2T^4 + \frac{1}{T^2})}\sum_{i=1}^n \int\limits_Q f_{1\,x_i}^2(x,t)\,dxdt$$
$$+\left(\frac{1}{2} + \lambda_0^2H_1^2T^2 + 2H_1^2\right)\int\limits_Q Z^2(t,u)\,dxdt. \tag{20}$$

From (20) it follows that

$$(1 - M_5)\left(\int\limits_Q u_{tttt}^2(x,t)\,dxdt + \int\limits_Q [\Delta u(x,t)]^2\,dxdt\right) + \varepsilon\lambda_0 \int\limits_Q [\Delta u_t(x,t)]^2\,dxdt$$
$$+\sum_{i=1}^n \int\limits_Q [1 + (c(x,t) - c_t(x,t)(\lambda_0 - t))]u_{x_i}^2(x,t)\,dxdt + \int\limits_Q \left(\frac{1}{2} - 2c(x,t)\right) u_{tt}^2(x,t)\,dxdt$$
$$+\int\limits_Q (c(x,t) - c_t(x,t)(\lambda_0 - t) - c_{tttt}(x,t) - M_6)u^2(x,t)\,dxdt$$

$$
\begin{gathered}
-\sum_{i=1}^{n}\int\limits_{Q} u_{x_i tt}^2(x,t)\,dxdt + 2\varepsilon\lambda_0\sum_{i=1}^{n}\int\limits_{Q} u_{x_i t}^2(x,t)\,dxdt \\
+(\lambda_0 - T)\sum_{i=1}^{n}\int\limits_{\Omega} u_{x_i}^2(x,T)\,dx + 4\int\limits_{Q} c_{tt}(x,t)u_t^2(x,t)\,dxdt \\
+(\varepsilon + (\lambda_0 - T))\sum_{i=1}^{n}\int\limits_{\Omega} u_{x_i tt}^2(x,T)\,dx + (\lambda_0 - T)\int\limits_{\Omega} u_{tt}^2(x,T)\,dx \\
\le \left(2 + \lambda_0^2 T^2\right)\int\limits_{Q} f_1^2(x,t)\,dxdt \\
+\frac{1}{2(H_2^2\lambda_0^2 + 2\lambda_0^2 c_1^2 T^4 + \frac{1}{T^2})}\sum_{i=1}^{n}\int\limits_{Q} f_{1\,x_i}^2(x,t)\,dxdt + M_6, \qquad (21)
\end{gathered}
$$

where the constant M_6 is defined by $h(x,t)$, $K(x,t)$, and $c(x,t)$, the domain Ω, and the numbers λ_0, T, and

$$
M_5 = M_0\left(\frac{1}{2} + \lambda_0^2 H_1^2 T^2 + 2H_1^2\right).
$$

Estimate (21) implies that we can pass to the limit as $\varepsilon \to 0$ in the family $\{u_\varepsilon(x,t)\}$ of solutions to (10_ε), (7), (8). The limit function belongs to V and is a solution to (6)–(8).

Demonstrate that $u(x,t)$ and $q(t)$ are a solution to Inverse Problem I. To this end, we multiply (1) by $K(x,t)$ and integrate the result over Ω. We obtain

$$
\psi_{tttt}(t) \equiv \frac{\partial^4}{\partial t^4}\left(\int\limits_{\Omega} K(x,t)u(x,t)\,dx\right) = 0. \qquad (22)
$$

Since the initial conditions of Inverse Problem I are homogeneous, we have $\psi(t) = 0$ for $t \in (0,T)$. The latter means that a solution $u(x,t)$ to the boundary value problem (6)–(8) satisfies the overdetermination condition (4). Since $u(x,t)$ and $q(t)$ belong to the required classes, we can conclude that these functions is a solution to Inverse Problem I.

The theorem is proven.

In the case of Inverse Problem 2, we can establish a similar result to that of Theorem 1.

Theorem 2. *Assume that*

$$
\begin{gathered}
h(x,t) \in C^1(\overline{Q}), \quad c(x,t) \in C^4(\overline{Q}), \quad K(x,t) \in C^4(\overline{Q}), \\
f(x,t), f_{x_i}(x,t) \in L_2(Q), \quad i = 1,\dots,n, \\
h_0(t) \ne 0, \quad t \in [0,T], \quad c(x,T) \ge 0
\end{gathered}
$$

and there exists λ_0 *such that* $\lambda_0 > T$, $[(\lambda_0 - t)c(x,t)]_t \le 0$ *for* $(x,t) \in \overline{Q}$, *and*

$$
M_0\left(\lambda_0^2 T^2\left(H_1^2 + \frac{H_2^2}{2(1 - \lambda_0^2 c_1^2 T^6)}\right) + 2H_1^2\right) < 1,
$$

Then there exist $u(x,t) \in V$ *and* $q(t)$ *in* $L_2[0,T]$ *solving Inverse Problem* II.

The proof of Theorem 2 is rather similar to that in Theorem 1.

Remark. We can take a general elliptic operator of the second order in (1) rather than the Laplace operator.

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S. S. Pavlov
North-Eastern Federal University, Institute of Mathematics, Yakutsk, Russia
`ststepmath@mail.ru`

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RECOVERING PARAMETERS IN BOUNDARY VALUE PROBLEMS FOR LINEAR PARABOLIC EQUATIONS OF FOURTH ORDER

L. A. Teleshova

Abstract. We prove the existence theorems of regular solutions to boundary value problems for parabolic equations of fourth order when either the boundary conditions or the right-hand side contain unknown parameters depending on time. The overdetermination conditions are integrals over a spatial domain or its boundary.

Keywords: parabolic equation, inverse problem, integral overdetermination condition, Fourier series, regular solution, solvability

1. Introduction

In the article we study the problems of finding solutions to fourth order parabolic equations when either the boundary conditions or the right-hand side are unknown. These problems can be treated as inverse problems, i.e., the problems of determining some parameters (coefficients) of the problem together with a solution itself. As a rule, it is assumed in these problems that the unknown parameters have a special form and the natural boundary conditions are complemented with some additional conditions called the overdetermination conditions.

In the present article we assume that the unknown parameters are functions of time. We employ integral overdetermination conditions in which some integral of a solution over a spatial domain or its boundary are equated to zero.

The article consists of two sections and an addendum. In the first section, we study inverse problems of determining the boundary data and, in the second section, inverse problems of determining the right-hand side. These sections are unified by a common approach.

Possible generalizations of the results obtained are pointed out in the addendum.

2. Recovering Boundary Conditions

Assume that Ω is a bounded domain in $\mathbb{R}^n$ with smooth (infinitely differentiable) boundary Γ, while Q is the cylinder $\Omega\times(0,T)$ of finite height T, and $S=\Gamma\times(0,T)$ is the lateral boundary of Q. Next, $c(x)$, $f(x,t)$, $h_1(x)$, $h_2(x)$, $K(x)$, and $N(x)$ are given functions defined for $x\in\overline{\Omega}$ and $t\in[0,T]$. Assume that (l_1,l_2) is one of the pairs of boundary operators, with either $l_1u=u$, $l_2u=\frac{\partial u}{\partial\nu}$, or $l_1u=u$, $l_2u=\Delta u$, or $l_1u=\frac{\partial u}{\partial\nu}$, $l_2u=\frac{\partial\Delta u}{\partial\nu}$ (here and in what follows $\nu=(\nu_1,\dots,\nu_n)$ is the inner normal to Γ at a point x, Δ is the Laplace operator in $x_1,\dots,x_n$).

Inverse Problem I. *Find $u(x,t)$, $q_1(t)$, and $q_2(t)$ such that $u(x,t)$ satisfies the equation*

$$u_t + \Delta^2 u + c(x)u = f(x,t), \tag{1}$$

in Q and the conditions

$$u(x,0) = 0, \quad x \in \Omega; \tag{2}$$

$$l_1 u(x,t)|_{(x,t)\in S} = q_1(t)h_1(x)|_{(x,t)\in S}, \quad l_2 u(x,t)|_{(x,t)\in S} = q_2(t)h_2(x)|_{(x,t)\in S}; \tag{3}$$

$$\int\limits_\Omega K(x)u(x,t)\,dx = 0, \quad \int\limits_\Omega N(x)u(x,t)\,dx = 0, \quad 0 < t < T. \tag{4}$$

Inverse problems of determining unknown boundary data and a solution were studied in various settings in dependence on the form of unknown data and overdetermination conditions. A series of results on solvability of such problems is presented in the monographs [1–3] and the articles [4–9].

On the other hand, the problems for different classes of nonstationary equations with conditions in the form of integrals of a solution (with a weight) over a spatial domain have intensively been studied since recently (see [10–19]). Mainly, these articles refer to second order parabolic and hyperbolic equations in the one-dimensional case. However, we can note that the problems with the integral conditions are treated in [8, 11, 19, 20] as inverse problems; but only some special case of Inverse Problem I are examined.

One more remark. The statement of the inverse problem I is close to that of [8], but the methods to those used in [20].

Proceed with an essential part of the article.

We expose now some formal constructions. Define $v_0(x,t)$ as a solution to (1) satisfying (2) and the condition

$$l_1 u(x,t)|_{(x,t)\in S} = l_2 u(x,t)|_{(x,t)\in S} = 0. \tag{3$'$}$$

Let $\tilde h_j(x)$, $j = 1, 2$, be solutions to the problems

$$\Delta^2 \tilde h_j + c(x)\tilde h_j = 0,$$

$$l_k \tilde h_j(x)|_{x\in\Gamma} = \delta_k^j h_j(x)|_{x\in\Gamma}, \quad j,k = 1,2$$

(δ_k^j stands for the Kronecker symbol).

A solution $u(x,t)$ to Inverse Problem I is representable as

$$u(x,t) = v_0(x,t) + V(x,t) + w(x,t),$$

with $V(x,t)$ of the form

$$V(x,t) = q_1(t)\tilde h_1(x) + q_2(t)\tilde h_2(x),$$

and $w(x,t)$ a solution to the problem

$$w_t + \Delta^2 w + c(x)w = -q_1'(t)\tilde h_1(x) - q_2'(t)\tilde h_2(x),$$

$$l_1 w(x,t)|_{(x,t)\in S} = l_2 w(x,t)|_{(x,t)\in S} = 0.$$

Let $\{w_k(x)\}_{k=1}^\infty$ be an orthonormal system (in the $L_2(\Omega)$ space) of the eigenfunctions of the problem

$$\Delta^2 w + c(x)w = \lambda w, \ x \in \Omega, \quad l_1 w(x)|_{x\in\Gamma} = l_2 w(x)|_{x\in\Gamma} = 0,$$

where λ_k, $k = 1, \dots,$ are the corresponding eigenvalues.

Consider the Fourier series of $\tilde{h}_j(x)$, $j = 1, 2$, with respect to the system $\{w_k(x)\}_{k=1}^{\infty}$

$$\tilde{h}_j(x) = \sum_{k=1}^{\infty} a_{jk} w_k(x).$$

The function $w(x,t)$ is also representable by its Fourier series as follows:

$$w(x,t) = \sum_{k=1}^{\infty} c_k(t) w_k(x), \tag{6}$$

where the unknowns $c_k(t)$ are solutions to the Cauchy problem

$$c_k'(t) + \lambda_k c_k(t) = -a_{1k} q_1'(t) - a_{2k} q_2'(t), \quad c_k(0) = 0.$$

Put

$$d_k(t) = c_k(t) + a_{1k} q_1(t) + a_{2k} q_2(t).$$

We require that $q_j(t)$, $j = 1, 2$, satisfy the condition

$$q_j(0) = 0$$

(note that this condition is equivalent the natural consistency condition for solutions to Inverse Problem I). In this case $d_k(t)$ is a solution to the Cauchy problem

$$d_k'(t) + \lambda_k d_k(t) = \lambda_k (a_{1k} q_1(t) + a_{2k} q_2(t)), \quad d_k(0) = 0.$$

With $d_k(t)$ in hand, we can find $c_k(t)$ as follows:

$$c_k(t) = \lambda_k \left[a_{1k} \int_0^t e^{-\lambda_k (t-\tau)} q_1(\tau)\, d\tau + a_{2k} \int_0^t e^{-\lambda_k (t-\tau)} q_2(\tau)\, d\tau \right] \\ - a_{1k} q_1(t) - a_{2k} q_2(t).$$

If $c_k(t)$ are known then we can obtain the following representation of $u(x,t)$ through the known quantities and the unknown coefficients $q_1(t)$ and $q_2(t)$:

$$u(x,t) = v_0(x,t) + q_1(t)\tilde{h}_1(x) + q_2(t)\tilde{h}_2(x) - \sum_{k=1}^{\infty} \left[a_{1k} q_1(t) + a_{2k} q_2(t) \right] w_k(x) \\ + \sum_{k=1}^{\infty} \lambda_k \left[a_{1k} \int_0^t e^{-\lambda_k (t-\tau)} q_1(\tau)\, d\tau + a_{2k} \int_0^t e^{-\lambda_k (t-\tau)} q_2(\tau)\, d\tau \right] w_k(x). \tag{7}$$

Introduce the notations

$$\psi_1(t) = \int_\Omega K(x) v_0(x,t)\, dx, \quad \psi_2(t) = \int_\Omega N(x) v_0(x,t)\, dx,$$

$$\alpha_{1k} = \int_\Omega K(x) w_k(x)\, dx, \quad \alpha_{2k} = \int_\Omega N(x) w_k(x)\, dx, \quad k = 1, 2, \dots$$

$$R_{i,j}(t) = \sum_{k=1}^{\infty} \lambda_k a_{jk} \alpha_{ik} e^{-\lambda_k t}, \quad i, j = 1, 2.$$

Multiply (7) by $K(x)$ and integrate it over Ω. Taking (4) into account, we infer

$$\psi_1(t) + \int_0^t R_{1,1}(t-\tau)q_1(\tau)\,d\tau + \int_0^t R_{1,2}(t-\tau)q_2(\tau)\,d\tau = 0. \tag{8}$$

Similarly, multiplying (7) by $N(x)$ and integrating, we arrive at the relation

$$\psi_2(t) + \int_0^t R_{2,1}(t-\tau)q_1(\tau)\,d\tau + \int_0^t R_{2,2}(t-\tau)q_2(\tau)\,d\tau = 0. \tag{9}$$

Equalities (8) and (9) give rise a system of Volterra integral equations of the first kind respectively for the functions $q_1(t)$ and $q_2(t)$, its solvability allows us to find a solution $u(x,t)$ to (1) satisfying (2)–(4).

Denote by $\mathbf{R}_0$ the matrix with entries $R_{i,j}(0)$, $i,j = 1,2$.

Theorem 1. *Assume that*

$$c(x) \in C(\overline{\Omega}), \quad c(x) \geq 0 \text{ for } x \in \overline{\Omega},$$

$$K(x) \in C(\overline{\Omega}), \quad N(x) \in C(\overline{\Omega}), \quad h_j(x) \in W_2^4(\Omega),\ j = 1,2;$$

$$\det \mathbf{R}_0 \neq 0;$$

and the series $\sum_{k=1}^{\infty} \lambda_k^p a_{jk}\alpha_{ik}$ *converge absolutely for* $i,j = 1,2$, $p = 1,2,3$.

Then, for every $f(x,t)$ *such that* $f(x,t) \in L_2(Q)$, $f_t(x,t) \in L_2(Q)$, *and* $f_{tt}(x,t) \in L_2(Q)$, *Inverse Problem I has a solution* $\{u(x,t), q_1(t), q_2(t)\}$ *such that* $u(x,t) \in W_2^{4,1}(Q)$ *and* $q_1(t) \in W_2^1([0,T])$, $q_2(t) \in W_2^1([0,T])$.

Proof. Consider the system of integral equations (8), (9). Using the standard passage from the integral Volterra equation of the first kind to an integral equation of the second kind (by differentiation, which is possible), it is easy to verify that under the conditions of the theorem the system (8), (9) has a solution $\{q_1(t), q_2(t)\}$, with $q_i(t) \in W_2^1([0,T])$, $i = 1,2$. Obviously, the function $u(x,t)$ defined as

$$u(x,t) = v_0(x,t) + V(x,t) + w(x,t),$$

where $V(x,t)$ and $w(x,t)$ are calculated through $q_1(t)$ and $q_2(t)$, is a solution to (1) such that $u(x,t) \in W_2^{4,1}(Q)$ and (2)–(4) hold.

The theorem is proven.

Make some remarks.

1. The convergence of $\sum_{k=1}^{\infty} \lambda_k^p a_{jk}\alpha_{ik}$ means that the numbers a_{ij} or α_{ij} (or their products) rapidly decrease. The latter holds for instance, if the functions $K(x)$ and $N(x)$ are smooth and vanish on Γ together with their derivatives of the corresponding order.

2. The functions $K(x)$, $N(x)$, $h_1(x)$, and $h_2(x)$ can depend on t. In contrast to the previous case the corresponding series turn into functional series and the conditions of uniform convergence arise; the remaining arguments are almost the same.

3. Recovering the Right-Hand Side

Let $f(x,t)$, $h(x,t)$, and $K(x)$ be given for $x \in \overline{\Omega}$ and $t \in [0,T]$. In this section we consider the equation

$$u_t + \Delta^2 u = f(x,t) + q(t)h(x,t). \tag{10}$$

Inverse Problem II. *Find $u(x,t)$ and $q(t)$ satisfying* (10) *in Q and such that $u(x,t)$ meets the homogeneous boundary conditions of the form*

$$\frac{\partial u}{\partial \nu}\bigg|_{(x,t)\in S} = \frac{\partial \Delta u}{\partial \nu}\bigg|_{(x,t)\in S} = 0, \tag{11}$$

the initial condition (2) *and the boundary integral overdetermination condition*

$$\int\limits_{\Gamma} K(x)u(x,t)ds_x = 0, \quad t \in (0,T). \tag{12}$$

Inverse Problem III. *Find $u(x,t)$ and $q(t)$, satisfying* (10) *in Q such that $u(x,t)$ meets* (2), (11) *and the interior integral overdetermination condition*

$$\int\limits_{\Omega} N(x)u(x,t)\,dx = 0, \quad t \in (0,T). \tag{13}$$

Note that Inverse Problem II for higher order parabolic equations was studied earlier only in [21]. Inverse Problem III is studied by many authors (see the monographs [22–24] and the articles [25, 26]. We study solvability of Inverse Problem III here in order to describe solvability conditions in other terms than those in the above-mentioned articles.

Again we present some formal constructions.

Write out the Fourier series of $h(x,t)$ and $f(x,t)$ in the system $\{w_k(x)\}_{k=1}^{\infty}$ as follows:

$$h(x,t) = \sum_{k=1}^{\infty} h_k(t)w_k(x), \quad f(x,t) = \sum_{k=1}^{\infty} f_k(t)w_k(x).$$

The function $u(x,t)$ is also representable by its Fourier series

$$u(x,t) = \sum_{k=1}^{\infty} c_k(t)w_k(x),$$

where the unknowns $c_k(t)$ are solutions to the Cauchy problem

$$c_k'(t) + \lambda_k c_k(t) = f_k(t) + q(t)h_k(t), \quad c_k(0) = 0. \tag{14}$$

Solving this problem, we find that

$$c_k(t) = p_k(t) + \int\limits_0^t q(\tau)h_k(\tau)e^{-\lambda_k(t-\tau)}\,d\tau.$$

Inserting $w_k(x)$ and $c_k(t)$ into the representation for $u(x,t)$, we conclude that

$$u(x,t) = \sum_{k=1}^{\infty} \Bigg[\int\limits_0^t f_k(\tau)e^{-\lambda_k(t-\tau)}\,d\tau + \int\limits_0^t q(\tau)h_k(\tau)e^{-\lambda_k(t-\tau)}\,d\tau \Bigg] w_k(x). \tag{15}$$

Introduce the notations

$$\bar{h}_k = \max_{[0,T]} |h_k(t)|, \quad p_k(t) = \int_0^t f_k(\tau) e^{-\lambda_k(t-\tau)}\, d\tau, \quad \bar{p}_k = \max_{[0,T]} |p_k(t)|,$$

$$b_k = \int_\Gamma N(x) w_k(x) ds_x, \quad \phi(t) = \sum_{k=1}^\infty b_k p_k(t), \quad G(t,\tau) = \sum_{k=1}^\infty b_k h_k(\tau) e^{-\lambda_k(t-\tau)}.$$

$$\beta_k = \int_\Omega N(x) w_k(x)\, dx, \quad \eta(t) = \sum_{k=1}^\infty \beta_k p_k(t), \quad H(t,\tau) = \sum_{k=1}^\infty \beta_k h_k(\tau) e^{-\lambda_k(t-\tau)}.$$

Multiply (15) by $K(x)$ and integrate over Γ. Taking (12) into account, we infer

$$\phi(t) + \int_0^t q(\tau) G(t,\tau)\, d\tau = 0. \tag{16}$$

Similarly, multiplying (15) by $N(x)$, integrating over Ω, and taking (13) into account, we find that

$$\eta(t) + \int_0^t q(\tau) H(t,\tau)\, d\tau = 0. \tag{17}$$

Equations (16) and (17) are integral Volterra equations of the first kind respectively $q(t)$ and their solvability allows us to find a function $u(x,t)$ that is a solution to (10) satisfying (2) and (11).

Theorem 2. *Assume that*

$$h(x,t) \in C(\bar{Q}), \quad G_t(t,\tau) \in C((0,T),(0,T)),$$
$$\phi(t) \in C^1([0,T]), \quad \phi'(t) \in L_2([0,T]);$$
$$\left| \sum_{k=1}^\infty b_k h_k(t) \right| \geq b_0 > 0 \text{ for } t \in [0,T];$$

and the series $\sum_{k=1}^\infty |b_k| \bar{h}_k$, $\sum_{k=1}^\infty |b_k| \bar{p}_k$, *and* $\sum_{k=1}^\infty |b_k| \bar{h}_k \lambda_k$ *converge absolutely.*

Then for every $f(x,t)$ *such that* $f(x,t) \in L_2(Q)$ *and* $f_t(x,t) \in L_2(Q)$, *there exists a solution* $\{u(x,t), q(t)\}$ *to Inverse Problem* II *such that* $u(x,t) \in W_2^{2,4}(Q)$ *and* $q(t) \in L_2((0,T))$.

Theorem 3. *Assume that*

$$h(x,t) \in C(\bar{Q}), \quad \eta(t) \in C^1([0,T]),$$
$$\eta'(t) \in L_2([0,T]), \quad H_t(t,\tau) \in C((0,T),(0,T));$$
$$\left| \sum_{k=1}^\infty \beta_k h_k(t) \right| \geq d_0 > 0 \text{ for } t \in [0,T],$$

and the series $\sum_{k=1}^\infty |\beta_k| \bar{h}_k$, $\sum_{k=1}^\infty |\beta_k| \bar{p}_k$, *and* $\sum_{k=1}^\infty |\beta_k| \bar{h}_k \lambda_k$ *converge absolutely.*

Then, for every function $f(x,t)$ *such that* $f(x,t) \in L_2(Q)$ *and* $f_t(x,t) \in L_2(Q)$, *there exists a solution решение* $\{u(x,t), q(t)\}$ *to Inverse Problem* III *such that* $u(x,t) \in W_2^{2,4}(Q)$ *and* $q(t) \in L_2((0,T))$.

The proofs of Theorems 2 and 3 are reduced to the proof of solvability of (15) and (16), respectively. Under the conditions of Theorems 1 and 2 these equations are solvable (see, for instance, [27]).

4. Addendum

1. The results similar to those in Theorems 1 and 2 are easy to establish for parabolic higher order equations, for example for the equations

$$u_t + (-1)^m \Delta^m u + c(x)u = f(x,t).$$

For these equations, we assume that the system $\{l_j\}_{j=1}^m$ of boundary operators with coefficients independent of the variable t is such that the problem

$$w_t + (-1)^m \Delta^m w + c(x)w = \lambda w,$$

$$l_j w(x)|_{x\in\Gamma} = 0, \quad j = 1, 2, \dots, m,$$

possesses a system $\{w_k\}_{k=1}^\infty$ of eigenfunctions complete in $W_2^{2m}(\Omega)$ and orthonormal in $L_2(\Omega)$ with the corresponding nonpositive eigenvalues λ_k, $k = 1, 2, \dots$.

Point out that the operator $lu = u$ must not be among the operators $l_1, l_2, \dots, l_m$ in the case of Inverse Problem II.

2. In all constructions the operator Δ can be replaced with an operator L of the form

$$L = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(b^{ij}(x)\frac{\partial}{\partial x_j}\right),$$

with smooth coefficients $b^{ij}(x)$ in $\overline{\Omega}$.

3. Together with Inverse Problems II and III, it is not difficult to study the problems with point overdeterminations, namely, with the condition $u(x_0, t) = 0$ for some $x_0 \in \Gamma$ or $x_0 \in \Omega$.

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L. A. Teleshova
Buryat State University
Institute of Mathematics and Informatics, Ulan-Ude, Russia
love_20_09@mail.ru

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RESURGENT FUNCTIONS AND SINGULAR ODES

J. R. Ecalle

Abstract. We propose a brief introduction to the resurgence theory and give several applications to the study of singular ODEs.

Keywords: resurgent function, ordinary differential equation

§ 1. Divergence and Resurgence

Introduction: singular ODEs. The topic of the present article is the natural divergence (i.e., the divergence of power series appearing in solving purely analytic problems) and ways of its overcoming: resurgent functions and alien derivation. Explain what it is about by the example of local analytic ODEs (ordinary differential equations), i.e., on the example of the germs of such ODEs near a singular point z_0. It is convenient to take ∞ but not 0 as a singular point. Everything depends on the nature of a complete solution $\widetilde{Y}(z,u)$ (that is, of a formal solution saturated by the parameters u) to our ODE.

The case when a complete formal solution contains only power series is trivial (from the local standpoint) because these series converge.

The opposite case, when there is no complete formal solution (or no formal solution whatever) is hopeless (again from the local standpoint) since there is nothing to sum there.

The intermediate, so-called singular, case, when a formal solution is a mixture of exponents and power series is interesting and accessible: the series usually diverge but can be summed.

Borel summation method. Consider a singular analytic ODE $E(z,Y)=0$ around ∞ and confine ourselves first to the monocritical case, i.e., to the case when there is a complete formal solution with expansion

$$\widetilde{Y}(z,u)=\widetilde{Y}_0(z)+\sum_{n\in\mathbb{N}^d} u^n e^{\lambda_n z}\widetilde{Y}_n(z) \quad (u=(u_1,\dots,u_d)) \tag{1}$$

with divergent power series

$$\widetilde{Y}_n(z)=\sum a_{n,k}z^{-k},$$

but with simple exponents $e^{\lambda_n z}$ (“monocriticality”). The problem consists in turning this formal solution into a real one. To this end, we must sum up each $\widetilde{Y}_n(z)$, i.e., turn it into an analytic germ $Y_{n,\theta}(z)$ defined in some sectorial neighborhood of ∞ with bisector $\arg z^{-1}=\theta$ and admitting $\widetilde{Y}_n(z)$ as an asymptotic series. However, the “tilde removal,” i.e., summation, is possible only indirectly, via the intermediate step $\widehat{Y}_n$:

$$\widetilde{Y}_n(z) \overset{\mathscr{B}=Borel}{\longrightarrow} \widehat{Y}_n(\zeta) \overset{\mathscr{L}=Laplace}{\longrightarrow} Y_{n,\theta}(z) \overset{asympt^{ly}}{\sim} \widetilde{Y}_n(z). \tag{2}$$

The Borel transform ($\mathscr{B}$) acts "termwise":

$$\mathscr{B} : z^{-\sigma} \mapsto \zeta^{\sigma-1}/\Gamma(\sigma) \ (\sigma \notin -\mathbb{N}); \quad z^n \mapsto \delta^{(n)} \ (n \in \mathbb{N}, \ \delta = Dirac), \tag{3}$$

and turns each series $\widetilde{\varphi}(z)$ of Gevrey type 1 into a series $\widehat{\varphi}(\zeta)$ with nonzero radius of convergence

$$\mathscr{B} : \widetilde{\varphi}(z) = \sum a_n z^{-n} \mapsto \widehat{\varphi}(\zeta) = \sum \frac{a_n \zeta^{n-1}}{(n-1)!}. \tag{4}$$

Moreover, if a series $\widetilde{\varphi}$ is of a "natural origin" then $\widehat{\varphi}$ usually admits an analytic extension along almost all axes from 0 to ∞ without analytic barriers with a discrete configuration of singular points ω_i and at most exponential growth near ∞, which makes it possible to apply to $\widehat{\varphi}$ to the Laplace transform ($\mathscr{L}$) formally inverse to $\mathscr{B}$:

$$\mathscr{L} : \widehat{\varphi} \mapsto \varphi_\theta \quad \text{c} \quad \varphi_\theta(z) := \int_0^{\infty . e^{i\theta}} \widehat{\varphi}(\zeta)\, e^{-\zeta\, z}\, d\zeta. \tag{5}$$

The Borel transform turns the usual product into the so-called convolution $*$, and derivation with respect to z becomes multiplication by ζ. The Laplace transform carries out the inverse conversions:

$$\mathscr{B} : \widetilde{\varphi}_1 \cdot \widetilde{\varphi}_2 \mapsto \widehat{\varphi}_1 * \widehat{\varphi}_2 \quad \text{c} \quad (\widehat{\varphi}_1 * \widehat{\varphi}_2)(\zeta) := \int_0^{\zeta} \widehat{\varphi}_1(\zeta_1) * \widehat{\varphi}_2(\zeta - \zeta_1)\, d\zeta_1, \tag{6}$$

$$\mathscr{B} : \partial_z \widetilde{\varphi}(z) \mapsto -\zeta \widehat{\varphi}(\zeta), \tag{7}$$

$$\mathscr{B} : \psi(z) = (\omega + \partial_z)\varphi(z) \Longrightarrow \widehat{\varphi}(\zeta) = (\omega - \zeta)^{-1} \widehat{\psi}(\zeta), \tag{8}$$

$$\mathscr{B} : \psi(z) = \varphi(z) - \varphi(z+1) \Longrightarrow \widehat{\varphi}(\zeta) = (1 - \exp(-\zeta))^{-1} \widehat{\psi}(\zeta). \tag{9}$$

Relations (8) and (9) show that a solution to differential or difference equations can create in the functions $\widehat{\varphi}$ not only simple poles but, under a second convolution, more complicated singularities. The general "Borel" summation scheme looks as follows:

$$\begin{array}{ccccccc} \widetilde{\varphi}(z) & & \cdots & & \varphi_\theta(z) & & z\text{-}plane\ (multiplication) \\ & \mathscr{B} \searrow & & \nearrow \mathscr{L} & & & \\ & & \widehat{\varphi}(\zeta) & & & & \zeta\text{-}plane\ (convolution) \end{array}$$

Any axis of integration $\arg \zeta = \theta$ can move exactly as long as the singular points ω_i allow, and to each regular sector of angular measure $\delta\theta$ in the ζ-plane there corresponds a regular sector of angular measure $\delta\theta + \pi$ in the z-plane.

This shows the main problems and difficulties we face.

Problem 1. *For integrating in the sense of Laplace, we must make sure that $\widehat{\varphi}(\zeta)$ has no analytic barriers and does not grow superexponentially as $\zeta \to \infty$ along the straight line axes.*

Problem 2. *For controlling this growth, we must be able to bound the convolution integrals occurring implicitly in the definition of the function $\widehat{\varphi}(\zeta)$.*

Problem 3. *We must find all singular points ω of $\widehat{\varphi}$ and find out the behavior of $\widehat{\varphi}$ at each such ω since these singularities are responsible for the divergence of the initial series $\widetilde{\varphi}$, and the (Stokes constants)* bear important information.

Difficulty 1. The possible presence of singular points for ω over $\mathbb{R}^+$.

Difficulty 2. A high branching of the Riemann surfaces.

Difficulty 3. The complexity of integration paths over which we must calculate the simple and n-fold convolution.

§ 2. Multiplicative Averaging

Physicists are bound to think that, in the presence of singular points for ω_i over $\mathbb{R}^+$, the series $\widetilde{\varphi}$ cannot be summed. However, this fails. One should simply apply a suitable averaging μ of separate branches of a multi-valued function:

$$\widetilde{\varphi}(z) \xrightarrow{\mathscr{B}} \widehat{\varphi}(\zeta) \xrightarrow{\mu} \mu\widehat{\varphi}(\zeta) \xrightarrow{\mathscr{L}} \varphi(z). \tag{10}$$

The averaging $\mu : \widehat{\varphi} \mapsto \mu\widehat{\varphi}$ is defined by weights $\mu^{\binom{\epsilon}{\omega}}$:

$$\mu\widehat{\varphi}(\zeta) := \sum_{\epsilon_i\in\{+,-\}} \mu^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}} \widehat{\varphi}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}}(\zeta) \quad \text{if } \omega_r < \zeta < \omega_{r+1}. \tag{11}$$

Here $\omega_1, \omega_2, \dots$ stand for singular points lying over $\mathbb{R}^+$ and $\widehat{\varphi}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}}$ denotes the parametrization of the function $\widehat{\varphi}$ over the interval $]\omega_i, \omega_{i+1}[$ that corresponds to the bypass of the next point ω_i from the right if $\epsilon_i = +$ or from the left if $\epsilon_i = -$. Moreover, μ must satisfy the following two main conditions:

(i) it must preserve reality for applications to the problems where only real solutions are acceptable. Hence, if a multivalued function $\widehat{\varphi}(\zeta)$ is real for small $\zeta > 0$ then the single-valued $\mu\,\widehat{\varphi}(\zeta)$ must remain real for all $\zeta > 0$;

(ii) it must commute with the convolution:

$$\mu(\widehat{\varphi}_1 * \widehat{\varphi}_1) \equiv (\mu\widehat{\varphi}_1) * (\mu\widehat{\varphi}_2), \tag{12}$$

for applications to nonlinear problems. In other words, in scheme (10), as well as the left and right arrows, the middle arrow must be not only linear but also a multiplicative homomorphism. Both these conditions are hardly compatible. For example, it is clear that the "half-sum"

$$\mu^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}} = \begin{cases} \frac{1}{2} & \text{if } \epsilon_1 = \epsilon_2 = \dots = \epsilon_r, \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

satisfies (i) but not (ii). Fortunately, suitable averagings satisfying all requirements exist. Here are two main examples:

The standard averaging. Its weights are given by a direct formula:

$$\mu^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}} := \frac{\Gamma(p+\frac{1}{2})\Gamma(q+\frac{1}{2})}{\Gamma(r+1)\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})} = \frac{(2\,p)!(2\,q)!}{4^{p+q}p!q!(p+q)!} \tag{14}$$

with

$$p := \sum_{\epsilon_i=+} 1, \quad q := \sum_{\epsilon_i=-} 1 \quad (p+q=r). \tag{15}$$

The "organic" averaging. Its weights are given by a recurrent formula:

$$\mu^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}} := \mu^{\binom{\epsilon_1\ ,\dots,\ \epsilon_{r-1}}{\omega_1\ ,\dots,\ \omega_{r-1}}} \frac{1}{2}\left(1 + \epsilon_{r-1}\,\epsilon_r\,\frac{\omega_{r-1}}{\omega_r}\right) \quad \text{with } \mu^{\binom{\pm}{\omega_1}} := \frac{1}{2}. \tag{16}$$

§ 3. Alien Derivation

Calculation of the convolution over SSR-paths. Let $\Omega = \{\omega_1, \omega_2, \underbrace{\dots\}}$ be a discrete subset in $\mathbb{C}$ and let $\widehat{\varphi}_1, \widehat{\varphi}_2$ be two analytic functions on $\mathscr{R} := \widetilde{\mathbb{C}\setminus\Omega}$. For small ζ, the convolution integral (6) is calculated over the segment $[0, \zeta]$, and, for remote ζ, it must be calculated over so-called *self-symmetrically reducible* (SSR) paths, i.e., over paths $\Gamma_* \subset \mathbb{C}\setminus\Omega$, that are not only symmetric with respect to their midpoint $\zeta/2$ but also continuously contractible to zero under the constant preservation of the self-symmetry. Excluding a countable set, every $\zeta \in \mathscr{R}$ is an endpoint of such an SSR-path but the problem is that, even for simple surfaces $\mathscr{R}$, SSR-paths rotate and get more complicated as their endpoint moves away from ζ to an extent that they are practically inapplicable.

THE DEFINITION OF Δ-OPERATORS. Instead of useless SSR-paths, we need the linear operators $\widehat{\Delta}_\omega$ with subscripts $\omega \in \widetilde{\mathbb{C} \setminus \{0\}}$, which describe in details the behavior of $\widehat{\varphi}(\zeta)$ near a singular point ω (more exactly, over it), and acting "by the Leibniz rule":

$$\widehat{\Delta}_\omega(\widehat{\varphi}_1 * \widehat{\varphi}_2) \equiv (\widehat{\Delta}_\omega \widehat{\varphi}_1) * \widehat{\varphi}_2 + \widehat{\varphi}_1 * (\widehat{\Delta}_\omega \widehat{\varphi}_2). \tag{17}$$

Their action is defined by the formula

$$\widehat{\Delta}_\omega \widehat{\varphi}(\zeta) := \sum_{\epsilon_i = \pm} \frac{\epsilon_r}{2\pi i} \delta^{\left(\begin{smallmatrix} \epsilon_1 & ,\dots, & \epsilon_r \\ \omega_1 & ,\dots, & \omega_r \end{smallmatrix}\right)} \widehat{\varphi}^{\left(\begin{smallmatrix} \epsilon_1 & ,\dots, & \epsilon_r \\ \omega_1 & ,\dots, & \omega_r \end{smallmatrix}\right)}(\zeta + \omega) \quad (\omega_r := \omega), \tag{18}$$

first for small $\zeta \in [0, \omega]$ and is then extended analytically. Here $\omega_1, \omega_2, \dots$ are singular points lying between 0 and $\omega_r := \omega$, and in order to guarantee (17), the weights δ must satisfy strict algebraic conditions. For the "standard" Δ-operators, the weights depend only on ϵ:

$$\delta^{\left(\begin{smallmatrix} \epsilon_1 & ,\dots, & \epsilon_r \\ \omega_1 & ,\dots, & \omega_r \end{smallmatrix}\right)} := \frac{p!q!}{(p+q+1)!} \quad \text{c } p := \sum_{\epsilon_i = +}^{1 \le i \le r-1} 1, \;\; p := \sum_{\epsilon_i = -}^{1 \le i \le r-1} 1, \tag{19}$$

and for the "organic" Δ-operators, they also depend on ω:

$$\delta^{\left(\begin{smallmatrix} \epsilon_1 & ,\dots, & \epsilon_r \\ \omega_1 & ,\dots, & \omega_r \end{smallmatrix}\right)} := \begin{cases} (\omega_{p+1} - \omega_p)/(2\,\omega_r) & \text{if } (\epsilon_1, \dots, \epsilon_r) = ((+)^p, (-)^q, \epsilon_r), \\ (\omega_{q+1} - \omega_q)/(2\,\omega_r) & \text{if } (\epsilon_1, ..., \epsilon_r) = ((-)^q, (+)^p, \epsilon_r), \\ 0 & \text{otherwise.} \end{cases}$$

In view of (17), the operators $\widehat{\Delta}_\omega$ are called *alien derivations*, and the function $\widehat{\Delta}_\omega \widehat{\varphi}$ is referred to as the *alien derivative of* $\widehat{\varphi}$.

PROPERTIES OF THE Δ-OPERATORS. Alongside $\widehat{\Delta}_\omega$, it is convenient to consider the operators Δ_ω and $\mathbf{\Delta}_\omega$, which act directly in the z-plane both at series $\widetilde{\varphi}$ and germs φ_θ:

$$\widehat{\Delta}_\omega \overset{\text{pull back}}{\Longrightarrow} \Delta_\omega = \mathscr{B}^{-1} \widehat{\Delta}_\omega \mathscr{B} \Longrightarrow \mathbf{\Delta}_\omega := e^{-\omega z} \Delta_\omega. \tag{20}$$

For these versions, the Leibniz rule takes the form

$$\widehat{\Delta}_\omega(\widehat{\varphi}_1 * \widehat{\varphi}_2) \equiv (\widehat{\Delta}_\omega \widehat{\varphi}_1) * \widehat{\varphi}_2 + \widehat{\varphi}_1 * (\widehat{\Delta}_\omega \widehat{\varphi}_2) \quad (\zeta\text{-}plane), \tag{21}$$

$$\Delta_\omega(\varphi_1 \cdot \varphi_2) \equiv (\Delta_\omega \varphi_1) \cdot \varphi_2 + \varphi_1 \cdot (\Delta_\omega \varphi_2) \quad (z\text{-}plane), \tag{22}$$

$$\mathbf{\Delta}_\omega(\varphi_1 \cdot \varphi_2) \equiv (\mathbf{\Delta}_\omega \varphi_1) \cdot \varphi_2 + \varphi_1 \cdot (\mathbf{\Delta}_\omega \varphi_2) \quad (z\text{-}plane). \tag{23}$$

Owing to the exponential factor, the "boldfaced," or invariant, operators $\mathbf{\Delta}_\omega$ commute with the ordinary derivation $\partial := \partial_z$:

$$[\widehat{\Delta}_\omega, \widehat{\partial}] = -\omega \widehat{\Delta}_\omega \Longrightarrow [\Delta_\omega, \partial] = -\omega \Delta_\omega \Longrightarrow [\mathbf{\Delta}_\omega, \partial] = 0. \tag{24}$$

The Δ-operators of a certain type, that is, of "standard" or "organic" type, freely generate the same Lie algebra $\mathbb{\Delta}$; i.e., for every

$$\mathbf{\Delta} := \sum \gamma^{\omega_1, \dots, \omega_r} [\mathbf{\Delta}_{\omega_r} \dots [\mathbf{\Delta}_{\omega_2}, \mathbf{\Delta}_{\omega_1}]]$$

there exists φ *such that* $\mathbf{\Delta}\varphi \neq 0$. Observe that, in many applications, the action of the operators Δ_ω depends not on ω as an element of $\mathbb{C}_\bullet := \widetilde{\mathbb{C}\setminus\{0\}}$ but only on its projection $\dot{\omega}$ to $\mathbb{C}$.

The algebra $\mathbb{RES}$ of resurgent functions. Without going into details, refer as *resurgent* "*functions*" to the series $\widetilde{\varphi}(z)$, functions $\widehat{\varphi}(\zeta)$, and germs $\varphi_\theta(z)$ satisfying the general scheme of § 1. For more simplicity, we usually denote these "functions" by $\varphi(z)$ without a tilde and θ, not forgetting that all assertions about them must be interpreted in the "three" models parallelly. The space $\mathbb{RES}$ of resurgent functions is closed not only under multiplication (under the usual product in the z-models) and under convolution (in a ζ-model) but also under alien derivation.

A first application: the resurgence equations. There are simple criteria that make it possible to predict whether a formal solution φ to a given ODE is a resurgent function with respect to the first variable z. Let $E = 0$ be such an equation. Using the Leibniz rule, it is easy to obtain in a purely formal manner new equations both for the ordinary derivative and the alien derivatives of φ:

$$E(z,\varphi) = 0 \Longrightarrow \begin{cases} E_*(z,\varphi,\partial\varphi) = 0 & \text{(linear homogeneous in } \partial\varphi\text{)}, \\ E_\omega(z,\varphi,\Delta_\omega\varphi) = 0 & \text{(linear homogeneous in } \Delta_\omega\varphi\text{)}. \end{cases}$$

The general solution to the equation $E_\omega = 0$ usually has the form

$$\Delta_\omega\, \varphi = A_\omega\, \varphi_\omega \tag{$*$}$$

or, less frequently,

$$\Delta_\omega \varphi = \sum_{j=1}^{j=s} A_{j,\omega}\varphi_{j,\omega}, \tag{$**$}$$

whereA_ω (or $A_{j,\omega}$) is a nontrivial (usually transcendent) scalar quantity (the Stokes constant) and φ_ω (or $\varphi_{j,\omega}$) is a simple power series purely formally deducible from $E_\omega = 0$ and hence from $E = 0$. Note that here, without any analysis, we establish an analytic fact because formulas ($*$), ($**$) give an analytic extension of $\widehat{\varphi}$ up to ω in the ζ-plane. In the close relationship between φ and $\Delta_\omega\varphi$ is the curious but still universal tendency of such functions to self-reproduce at their singular points, at each of them! This explains the term "resurgence."

Getting rid of SSR-paths and overcoming "multi-valuedness." The action of the "broken" ω-shifts $\widehat{T}_\Gamma$ and the $\widehat{\Delta}_\Gamma$-operators is given by the formulas

$$\widehat{T}_\Gamma\widehat{\varphi}(\zeta) := \widehat{\varphi}_\Gamma(\zeta+\omega), \quad \widehat{\Delta}_\Gamma\widehat{\varphi}(\zeta) := \widehat{\varphi}_\Gamma^+(\zeta+\omega) - \widehat{\varphi}_\Gamma^-(\zeta+\omega) \quad (\omega = \text{the end of } \Gamma)$$

first for small ζ, and then in the large, by analytic extension along a finite finitely punctured broken line Γ with a prescription for bypassing each of the punctured points. These "broken" operators can be represented uniquely as a polynomial of finitely many $\widehat{\Delta}$-operators and the rotation $R := \widehat{\varphi}(\zeta) \mapsto \widehat{\varphi}(e^{2\pi i}\zeta)$ ($\zeta \in \mathbb{C}_\bullet := \widetilde{\mathbb{C}\setminus\{0\}}$):

$$\widehat{T}_\Gamma = \mathrm{id} + \sum_r \sum_n \sum_{\omega_i} (2\pi i)^r \tau^{\omega_1,\dots,\omega_r} R^n \widehat{\mathbf{\Delta}}_{\omega_r} \dots \widehat{\mathbf{\Delta}}_{\omega_1} \quad (\tau^{\boldsymbol{\omega}} \in \mathbb{Q},\ n \in \mathbb{Z}), \tag{26}$$

$$\widehat{\Delta}_\Gamma = \widehat{\mathbf{\Delta}}_\omega + \sum_r \sum_n \sum_{\omega_i} (2\pi i)^r \lambda^{\omega_1,\dots,\omega_r} R^n \widehat{\mathbf{\Delta}}_{\omega_r} \dots \widehat{\mathbf{\Delta}}_{\omega_1} \quad (\lambda^{\boldsymbol{\omega}} \in \mathbb{Q}, \sum \omega_i = \omega).$$

This representation is extremely convenient since it reduces all operations of the type $(\widehat{\varphi}_1, \widehat{\varphi}_2) \mapsto \widehat{\varphi}_1 * \widehat{\varphi}_2$ or of the type $\widehat{\varphi} \mapsto (\widehat{\varphi})^{*n}$ to operations on the first leaf of a Riemann surface, i.e., on the common "analyticity star."

As we see, the Δ-operators deliver us from

(i) multivalued functions $\widehat{\varphi}$;

(ii) complicated multisheet Riemannian surfaces,

(iii) impossibly winding SSR-paths of integration.

Owing to the Δ-operators, it suffices to consider the functions $\widehat{\varphi}$ and all their alien derivatives as single-valued functions on their common analyticity star (centered at $0_\bullet$), and then all operations are reduced to ordinary operations over single-valued functions.

A survey of the main notions and principal rules.

THE PRIMARY Δ-OPERATORS (alien derivations):

$$\widehat{\Delta}_\omega \text{ in the } \zeta\text{-plane} \Longrightarrow \Delta_\omega \text{ in the } z\text{-}plane \text{ (“pull back”)} \tag{27}$$

THE SECONDARY $\boldsymbol{\Delta}$-OPERATORS: $\boldsymbol{\Delta}_\omega := e^{-\omega\, z}\Delta_\omega$ in the z-plane:

$$[\partial_z, \boldsymbol{\Delta}_\omega] \equiv 0, \quad \boldsymbol{\Delta}_\omega(f \circ g)(z) \equiv (\boldsymbol{\Delta}_\omega f) \circ g(z) \text{ in } g(z) \sim z. \tag{28}$$

THE $\mathbf{Z}$-SYMBOLS $\mathbf{Z}^{\boldsymbol{\omega}} = \mathbf{Z}^{\omega_1,\dots,\omega_r}$ (pseudovariables). This notation is dual to the $\boldsymbol{\Delta}$-operators. The product of $\mathbf{Z}$-symbols is given by "shuffling" the indices:

$$\partial_z \mathbf{Z}^{\boldsymbol{\omega}} \equiv 0, \quad \mathbf{Z}^{\boldsymbol{\omega}} \circ g \equiv \mathbf{Z}^{\boldsymbol{\omega}}, \quad \mathbf{Z}^{\boldsymbol{\omega}'}\mathbf{Z}^{\boldsymbol{\omega}''} = \sum \mathbf{Z}^{\boldsymbol{\omega}} \quad (\textit{shuffle product}). \tag{29}$$

THE DISPLAY. This is a kind of an "alien Taylor series":

$$\operatorname{dpl} \varphi := \varphi + \sum_r \sum_{\omega_j} \mathbf{Z}^{\omega_1,\dots,\omega_r} \boldsymbol{\Delta}_{\omega_r} \dots \boldsymbol{\Delta}_{\omega_1} \varphi. \tag{30}$$

It has both local nature (the z-part) and global nature (the $\mathbf{Z}$-part). It encodes in an extremely concise and convenient form the information about a function $\widehat{\varphi}(\zeta)$ on all leaves of its Riemannian surface. The main property of displays is the automatic extension of any relation between functions to a relation between displays:

$$R(\varphi_1, \dots, \varphi_s) \equiv 0 \Longrightarrow R(\operatorname{dpl} \varphi_1, \dots, \operatorname{dpl} \varphi_s) \equiv 0. \tag{31}$$

§ 4. Singular ODEs and the Bridge Equation

The *bridge equation* owes its name to the fact that it connects the alien and usual derivatives. It has a vast field of applications. Let us first point out the main facts about it and then give several examples. Its looks as follows:

$$\boldsymbol{\Delta}_\omega Y(z,u) = \mathbf{A}_\omega Y(z,u) \quad (\forall \omega \in \Omega) \tag{32}$$

• $Y(z,u)$ is a formal solution to a singular ODE with maximal number of parameters $u := (u_1, \dots, u_d)$. We omit the tilde for simplicity.

• $\boldsymbol{\Delta}_\omega$ is the "alien derivation." The subscript ω ranges over a countable set $\Omega \subset \mathbb{C} \setminus \{0\}$ or $\Omega \subset \widetilde{\mathbb{C} \setminus \{0\}}$.

• $\mathbf{A}_\omega$ is a "Stokes operator." This is an ordinary differential operator with respect to z and the parameters $u_1, \dots, u_d$. Its form is the "most general of all formally admissible forms."

• $\{\mathbf{A}_\omega; \omega \in \Omega\}$ is a complete system of Stokes constants.

• The whole divergence of $Y(z,u)$ is concentrated in power series with respect to z^{-1}.

The bridge equation and the display. Since the bridge equation can be iterated:

$$\begin{aligned}\mathbf{\Delta}_{\omega_1} Y(z,u) &= \mathbf{A}_{\omega_1} Y(z,u) \Longrightarrow\\ \mathbf{\Delta}_{\omega_2}\mathbf{\Delta}_{\omega_1} Y(z,u) &= \mathbf{\Delta}_{\omega_2}\mathbf{A}_{\omega_1} Y(z,u)\\ &= \mathbf{A}_{\omega_1}\mathbf{\Delta}_{\omega_2} Y(z,u)\\ &= \mathbf{A}_{\omega_1}\mathbf{A}_{\omega_2} Y(z,u) \text{ (order reversal!)},\end{aligned}$$

it immediately gives the alien derivatives of all orders:

$$\mathbf{\Delta}_{\omega_r}\dots\mathbf{\Delta}_{\omega_1} Y(z,u) = \mathbf{A}_{\omega_1}\dots\mathbf{A}_{\omega_r} Y(z,u). \tag{33}$$

Involving (33) in the definition of the display (30), we get

$$\begin{aligned}\operatorname{dpl} Y(z,u) &= Y(z,u) + \sum_r\sum_{\omega_i} \mathbf{Z}^{\omega_1,\dots,\omega_r}\mathbf{\Delta}_{\omega_r}\dots\mathbf{\Delta}_{\omega_1} Y(z,y)\\ &= Y(z,u) + \sum_r\sum_{\omega_i} \mathbf{Z}^{\omega_1,\dots,\omega_r}\mathbf{A}_{\omega_1}\dots\mathbf{A}_{\omega_r} Y(z,y).\end{aligned} \tag{34}$$

Riccati singular equations are the simplest nonlinear ODEs:

$$Y' = Y + H^-(z) + H^+(z)Y^2 \quad (H^\pm(z) \in z^{-1}\,\mathbb{C}\{z^{-1}\}). \tag{35}$$

The general solution can be written down in inhomogeneous or homogeneous form with the use of S_i or T_i. The former have infinitely many singular points in the ζ-plane, and the latter have only two:

$$Y(z,u) = \frac{ue^z S_0 + S_-}{ue^z S_0 S_+ + 1} = \frac{ue^z T_1 + T_2}{ue^z T_3 + T_4}, \quad \det\begin{pmatrix} T_1 & T_2\\ T_3 & T_4\end{pmatrix} = 1, \tag{36}$$

$$\begin{gathered}\widehat{S}_\pm(\zeta)\,\text{sing. over}\,\pm\beta N^* \quad \| \quad \widehat{T}_1,\widehat{T}_3\,\text{sing. over}\{0,1\},\\ \widehat{S}_0(\zeta)\,\text{sing. over }\ \mathbb{Z} \quad \| \quad \widehat{T}_2,\widehat{T}_4\ \text{ sing. over}\{0,-1\}.\end{gathered}$$

In this case, the bridge equation has a rather simple form:

$$\mathbf{\Delta}_{\pm 1} Y(z,u) = \mathbf{A}_{\pm 1} Y(z,u) \text{ c } \mathbf{A}_{\pm 1} = \alpha_{\pm 1} u^{1\pm 1}\partial_u. \tag{37}$$

Hence, we obtain the following resurgence equations for separate components:

$$\begin{gathered}\Delta_{+1}T_1 = \alpha_1 T_2, \quad \Delta_{+1}T_2 = 0, \quad \Delta_{+1}T_3 = \alpha_1 T_4, \quad \Delta_{+1}T_4 = 0,\\ \Delta_{-1}T_2 = \alpha_{-1}T_1, \quad \Delta_{-1}T_1 = 0, \quad \Delta_{-1}T_4 = \alpha_{-1}\,T_3, \quad \Delta_{-1}T_3 = 0,\\ \Delta_{+1}S_0 = a_1 S_-, \quad \Delta_{+1}S_+ = a_1 S_0^{-1}(1 - S_+S_-), \quad \Delta_{+1}\,S_- = 0,\\ \Delta_{-1}S_0 = a_{-1}S_+, \quad \Delta_{-1}S_- = a_{-1}S_0(1 - S_+S_-), \quad \Delta_{-1}S_+ = 0.\end{gathered}$$

The corresponding displays, with their clear separation of the z-variable and the $\mathbb{Z}$-symbols, carefully algebrize the complicated geometry of the ζ-plane:

$$\begin{pmatrix}\operatorname{dpl} T_1 & \operatorname{dpl} T_2\\ \operatorname{dpl} T_3 & \operatorname{dpl} T_4\end{pmatrix} = \begin{pmatrix} T_1 & T_2\\ T_3 & T_4\end{pmatrix} \times \begin{pmatrix}\mathbf{T}_1 & \mathbf{T}_2\\ \mathbf{T}_2 & \mathbf{T}_4\end{pmatrix}, \tag{38}$$

$$\operatorname{dpl} S_0 = \frac{S_0\mathbf{S}_0 + S_-\mathbf{S}_0\mathbf{S}_+}{1 + S_0S_+\mathbf{S}_-}, \quad \operatorname{dpl} S_\pm = \frac{S_\pm + S_0^{\mp 1}\mathbf{S}_\pm}{1 + S_0^{\mp 1}\,S_\mp\mathbf{S}_\pm}, \tag{39}$$

$$\mathbf{T}_1 = \mathbf{1} + \sum(\alpha_1\alpha_{-1})^n \mathbf{Z}^{\{1,-1\}^n}, \quad \mathbf{T}_2 = \sum \alpha_{-1}(\alpha_1\alpha_{-1})^n \mathbf{Z}^{-1,\{1,-1\}^n},$$

$$\mathbf{T}_3 = \sum \alpha_1(\alpha_{-1}\alpha_1)^n \mathbf{Z}^{1,\{-1,1\}^n}, \mathbf{T}_4 = \mathbf{1} + \sum(\alpha_{-1}\alpha_1)^n \mathbf{Z}^{\{-1,+1\}^n},$$

$$\log \mathbf{S}_0 = \sum_{1\le r} \sum_{\epsilon_1+\delta+\epsilon_r=0}^{\epsilon_i=\pm 1} \epsilon_1 \gamma_{\epsilon_1,\dots,\epsilon_r} \alpha_{\epsilon_1} \dots \alpha_{\epsilon_r} \mathbf{Z}^{\epsilon_1,\dots,\epsilon_r},$$

$$\mathbf{S}_\pm = \pm \sum_{2\le r} \sum_{\epsilon_1+\dots+\epsilon_r=\pm 1}^{\epsilon_i=\pm 1} \gamma_{\epsilon_1,\dots,\epsilon_r} \alpha_{\epsilon_1} \dots \alpha_{\epsilon_r} \mathbf{Z}^{\epsilon_1,\dots,\epsilon_r};$$

moreover,

$$\gamma_{\epsilon_1,\dots,\epsilon_r} := \prod \epsilon_j \prod(\epsilon_1 + \dots + \epsilon_r).$$

Starting from these formulas, it is interesting to follow how the relations between the components of two types

$$T_1T_4 - T_2T_3 = 1, \quad Y_0 = T_1/T_4, \quad Y_+ = T_3/T_1, \quad Y_- = T_2/T_4, \tag{40}$$

$$T_1^2 \equiv Y_0(1 - Y_+Y_-)^{-1}, \quad T_2^2 \equiv Y_0^{-1}Y_-^2(1 - Y_+Y_-)^{-1}, \tag{41}$$

$$T_3^2 \equiv Y_0Y_-^2(1 - Y_+Y_-)^{-1}, \quad T_4^2 \equiv Y_0^{-1}(1 - Y_+Y_-)^{-1} \tag{42}$$

automatically yield relations between displays:

$$R(T_i, Y_j) \equiv 0 \Longrightarrow R(\operatorname{dpl} T_i, \operatorname{dpl} Y_j) \equiv 0 \Longrightarrow R(\mathbf{T}_i, \mathbf{Y}_j) \equiv 0.$$

For checking the relations $R(\mathbf{T}_i, \mathbf{Y}_j) \equiv 0$, we can put $\alpha_1 = \alpha_{-1} = 1$ and apply the multiplication rule (29) to the series of $\mathbf{Z}$-symbols:

$$\mathbf{T}_1 = 1 + \mathbf{Z}^{+-} + \mathbf{Z}^{+-+-} \dots, \quad \mathbf{T}_2 = \mathbf{Z}^{-} + \mathbf{Z}^{-+-} + \mathbf{Z}^{-+-+-} \dots,$$

$$\mathbf{T}_3 = \mathbf{Z}^{+} + \mathbf{Z}^{+-+} + \mathbf{Z}^{+-+-+} \dots, \quad \mathbf{T}_4 = 1 + \mathbf{Z}^{-+} + \mathbf{Z}^{-+-+} \dots,$$

$$\mathbf{Y}_+ = \mathbf{Z}^{+} - 2\mathbf{Z}^{+-+} + 4\mathbf{Z}^{++-+-} + 12\mathbf{Z}^{+++--} + \cdots,$$

$$\mathbf{Y}_- = \mathbf{Z}^{-} - 2\mathbf{Z}^{-+-} + 4\mathbf{Z}^{--+-+} + 12\mathbf{Z}^{---++} + \cdots,$$

$$\log \mathbf{Y}_0 = (\mathbf{Z}^{+-} - \mathbf{Z}^{-+}) - 2(\mathbf{Z}^{++--} - \mathbf{Z}^{--++}) + 4(\mathbf{Z}^{++-+-} - \mathbf{Z}^{--+-+}) - 12(\mathbf{Z}^{+++--} - \mathbf{Z}^{---++}) \dots.$$

First-order singular ODEs. Pass to the general ODE formally adjoint to the equation $Y' = Y$:

$$Y' = Y + \sum_{0\le n} H_n(z)Y^n \quad (\sum H_nY^n \in z^{-1}\mathbb{C}\{z^{-1}, Y\}). \tag{44}$$

The complete solution involves exponents and divergent series Y_m:

$$Y(z,u) = Y_0(z) + \sum_{1\le m} u^m e^{m\,z} Y_m(z) \quad (Y_m(z) \in \mathbb{C}[[z^{-1}]]). \tag{45}$$

Each $Y_m(z)$ is a resurgent function, and the singular points of $\widehat{Y}_m(\zeta)$ lie over $m \in \mathbb{N}$. Here the bridge equation takes the form

$$\mathbf{\Delta}_n Y(z,u) = \mathbf{A}_n Y(z,u) \text{ c } \mathbf{A}_n = a_n\, u^{n+1}\partial_u \quad (\forall n \in \{-1\} \cup \mathbb{N}^*), \tag{46}$$

which leads to separate resurgence equations

$$\Delta_n Y_m = (m-n)a_n Y_{m-n} \quad (m \in \mathbb{N},\ -1 \le n \le m,\ a_n \in \mathbb{C}). \tag{47}$$

The special equations

$$\Delta_{-1} Y_0 = a_{-1}\, Y_1,\ \Delta_{-1} Y_1 = 2a_{-1} Y_2,\ \dots, \Delta_{-1} Y_{m-1} = m a_{-1} Y_m$$

imply that $(\Delta_{-1})^m Y_0 = m!(a_{-1})^m Y_m$. This means that (provided that the key coefficient $a_{-1} \neq 0$) the whole sequence $Y_1, Y_2, Y_3 \dots$ can be constructively deduced from known Y_0, which is of course impossible for regular ODEs. This shows that, on contrast to regular ODEs, singular ODEs are remarkably connected: knowing even a small part of their solution, one can reconstruct the whole solution.

For comparison: if $P(x)$ is an irreducible (reducible) polynomial over $\mathbb{Q}$ then it is possible (not possible) to deduce all roots from one root x_0.

Singular differential systems. Results very close to those given above are available:

(i) for an ODE of order $d \ge 2$ with dihedral Newton polygon;

(ii) for nonautonomous differential systems:

$$Y_j' = \lambda_j Y_j + h_j(z, Y_1, \dots, Y_n) \quad (1 \le j \le n), \tag{48}$$

(iii) for autonomous differential systems ("vector fields"):

$$Y_j' = \lambda_j Y_j + h_j(Y_1, \dots, Y_n) \quad (1 \le j \le n) \tag{49}$$

with single resonance $\sum m_j \lambda_j = \lambda_{j_0}$ $(m_j \ge 0)$.

§ 5. Transcendence. Analysis and Synthesis.

Transcendence and independence theorems. The display facilitates proving the theorems on the independence of solutions $\varphi_i(z) = \sum a_{j,n} z^{-n}$ to separate ODEs, i.e., proving the fact that such φ_i are in general not connected by any new relations. And the reason is simple: the embedding of 2-symbols into a hypothetical relation R imposes a huge number of new, very hardly fulfillable conditions:

$$R(\varphi_1, \dots, \varphi_s) = 0 \Longrightarrow R(\operatorname{dpl}\varphi_1, \dots, \operatorname{dpl}\varphi_s) = \sum^{p,q,\omega_i} R_{p,\omega_1,\dots,\omega_q} \overbrace{z^{-p}\mathbb{Z}^{\omega_1,\dots,\omega_q}}^{p+q=:N} = 0.$$

Indeed, when $N := p+q \to \infty$, the number of the relations $R_{p,\omega_1,\dots,\omega_q} = 0$ grows as $\mathscr{O}(N^{1+k})$, and the number of their coefficients $a_{j,n}$ grows only as $\mathscr{O}(s.N)$, which easily leads us to a contrdiction.

In this context, **analysis** is the problem of describing the Stokes constants A_ω. Here the two approaches are possible: a computational approach and a theoretical approach.

For the dominant constants A_ω (ω on the boundary of the convergence disk), starting from $\widehat{\Delta}_\omega \widehat{\varphi}(\zeta) := A_\omega \widehat{\varphi}_\omega(\zeta)$, where $\widehat{\varphi}$ and $\widehat{\varphi}_\omega$ are well known, one can effectively compute A_ω by means of the asymptotic analysis of the coefficients $\widehat{\varphi}$. For the nondominant Stokes constants A_ω, we must first take the point ω into the convergence disk by a conformal mapping.

Now, let us discuss the theoretical approach. Expanding a solution φ in the series

$$\varphi(z) = \Big(\sum_r \sum_{\omega_i} \boldsymbol{W}^{\omega_1,\dots,\omega_r}(z)\mathbf{B}_{\omega_r}\dots\mathbf{B}_{\omega_1}\Big) z, \tag{50}$$

deduce the explicit expression for the Stokes constants:

$$\mathbf{A}_{\omega_0} = \Big(\sum_r \sum_{\omega_1+\dots+\omega_r=\omega_0} W^{\omega_1,\dots,\omega_r}(z)\mathbf{B}_{\omega_r}\dots\mathbf{B}_{\omega_1}\Big)z, \tag{51}$$

where $\mathbf{B}_\omega$ are simple differential operators encoding the Taylor coefficients of our ODE and $\boldsymbol{W}^{\omega_1,\dots,\omega_r}$ are elementary resurgent functions with elementary behavior under the $\boldsymbol{\Delta}$-derivation:

$$\boldsymbol{\Delta}_{\omega_0}\boldsymbol{W}^{\omega_1,\dots,\omega_r}(z) = \sum_{1\le j\le r}\sum_{\omega_1+\dots+\omega_j=\omega_0} W^{\omega_1,\dots,\omega_j}\boldsymbol{W}^{\omega_{j+1},\dots,\omega_r}(z). \tag{52}$$

The elementary functions $\boldsymbol{W}^{\boldsymbol{\omega}}(z)$ are called the *resurgent monomials*, and the numbers $W^{\boldsymbol{\omega}}$ are known as the *resurgent monics*. The resurgent monics are similar to hyperlogarithms in their form.

Synthesis is the problem inverse to "analysis": for a certain type of a singular ODE, it is required to construct by explicit formulas the more "natural" or the "simplest" ODE of a given form possessing a prescribed set of Stokes constant: $\{\mathbf{A}_\omega, \omega\in\Omega\} \Longrightarrow$ ODE.

The synthesis relies upon resurgent monomials $\boldsymbol{U}_c^{\boldsymbol{\omega}}(z)$ different from the monomials $\boldsymbol{W}_c^{\boldsymbol{\omega}}(z)$ used in the analysis. Here is their definition:

$$\boldsymbol{U}_c^{\omega_1,\dots,\omega_r}(z) := \mathrm{SPA}\int_0^\infty \frac{e^{\sum_j \omega_j(z-y_j)+c^2\sum_j\bar{\omega}_j(z^{-1}-y_j^{-1})}\,dy_1\dots dy_r}{(y_r-y_{r-1})\dots(y_2-y_1)(y_1-z)}, \tag{53}$$

where SPA is the standard averaging of the integration paths. List the main properties of these new "monomials":

$$\boldsymbol{U}_c^{\boldsymbol{\omega}'}\boldsymbol{U}_c^{\boldsymbol{\omega}''} = \sum^{\boldsymbol{\omega}\in\mathrm{sha}(\boldsymbol{\omega}',\boldsymbol{\omega}'')} \boldsymbol{U}_c^{\boldsymbol{\omega}} \quad \text{("shuffle product")}, \tag{54}$$

$$\boldsymbol{\Delta}_{\omega_0}\boldsymbol{U}_c^{\omega_1,\dots,\omega_r} = \boldsymbol{U}_c^{\omega_2,\dots,\omega_r} \text{ if } \omega_0=\omega_1 \quad (\text{respectively } = 0 \text{ if } \omega_0=\omega_1), \tag{55}$$

$$\partial_z\boldsymbol{U}_c^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega}'\cdot\omega''=\omega} \boldsymbol{U}_c^{\boldsymbol{\omega}'}\mathbf{U}_c^{\boldsymbol{\omega}''} = \text{ earlier monomials } \boldsymbol{U}_c^{\boldsymbol{\omega}'}. \tag{56}$$

One should also observe the similatity of the behaviors of $\boldsymbol{U}_c^{\boldsymbol{\omega}}$ at the antipodes $z=\infty$ and $z=0$. Now, pass to the general scheme of the synthesis. For each $c>0$, it is easy to find the (unique) formal solution in the form of the Stokes constants $\mathbf{A}_\omega$ on the one side and the "monomials" $\boldsymbol{U}_c^{\boldsymbol{\omega}}$ on the other. For example, for resonance vector fields X, the solution is as follows: $X=\Theta\partial\Theta^{-1}$, where

$$\Theta := 1+\sum(-1)^r\boldsymbol{U}_c^{\omega_1,\dots,\omega_r}\mathbf{A}_{\omega_r}\dots\mathbf{A}_{\omega_1}, \tag{57}$$

$$\Theta^{-1} := 1+\sum\boldsymbol{U}_c^{\omega_1,\dots,\omega_r}\mathbf{A}_{\omega_1}\dots\mathbf{A}_{\omega_r}. \tag{58}$$

The convergence of these series (which is important) is guaranteed automatically for sufficiently large c. We should also mention the following curious phenomenon: though the constructed ODE is defined near $z=\infty$, by the antipodality of our $\boldsymbol{U}^{\boldsymbol{w}}$, to it there corresponds another ODE, defined near $z=0$ (its "antipodal shadow").

§ 6. Multicritical ODEs and Acceleration

When the complete solution to a given ODE is a series in z^{-1} and various elementary blocks $u_i e^{\sigma_{ij} z_j}$ and also $z_1 \prec z_2 \prec \dots \prec z_r$ (for example, $z_j \equiv z^{\alpha_j}$ and $0 < \alpha_j \uparrow$)), the summation scheme gets more complicated. For example, this applies to ODEs with more than one dihedral Newton polygon and to vector fields with multiple resonance. One must pass through several Borel planes: as many as there are "critical times" z_j. The transitions $\widehat{\varphi}_j(\zeta_j) \to \widehat{\varphi}_{j+1}(\zeta_{j+1})$ are carried out by the so-called acceleration integrals $\mathscr{C}_{j,j+1}$ but the situation remains the same in each of the ζ_j-planes: at each function $\widehat{\varphi}_j(\zeta_j)$, there act its own $\mathbf{\Delta}$-operators, generating their own resurgence equations with their own Stokes constants $\mathbf{A}_\omega$. The general scheme is as follows:

$$
\begin{array}{ccccccc}
\widetilde{\varphi}_1(z_1) & \leftarrow & \widetilde{\varphi}(z) & & \varphi(z) & \leftarrow & \varphi_r(z_r) \\
\downarrow \mathscr{B} & & & & & & \mathscr{L} \uparrow \\
\widehat{\varphi}_1(\zeta_1) & \underset{\mathscr{C}_{1,2}}{\rightarrow} & \widehat{\varphi}_2(\zeta_2) & \underset{\mathscr{C}_{2,3}}{\rightarrow \cdots \rightarrow} & \widehat{\varphi}_{r-1}(\zeta_{r-1}) & \underset{\mathscr{C}_{r-1,r}}{\rightarrow} & \widehat{\varphi}_r(\zeta_r)
\end{array}
$$

§ 7. Conclusion

In mathematics, resurgent functions are also applied:

(i) in the so-called difference equations;

(ii) in various functional equations; for example, in delay ODEs;

(iii) in discrete dynamical systems;

(iv) in singular perturbations, i.e., in expansions in a regular parameter ϵ; for example, in ODEs of the type

$$E(z, \varphi, \varphi', \dots, \varphi^{(d-1)}) + \epsilon\, \varphi^d = 0;$$

(v) in partial differential equations (O. Costin);
and many other problems.

In physics, resurgent functions are more and more frequent in expansions:

(i) in the Planck constant $\hbar$, in the so-called quasiclassical approach to quantum mechanics;

(ii) in the gauge coupling parameter α;

(iii) and in seemingly all important "small" constants of physics since, in accordance with Berry's principle: "When the nonvanishing of some small physical parameter ϵ means passing from a classical theory to its nonclassical generalization then the expansions in power series in ϵ as a rule diverge, which reflects the nontriviality of this passage" (See the recent conference on resurgence and string theory held in CERN, Geneva from June 29 to July 3, 2014). In conclusion, we stress that:

(i) the $\mathbf{\Delta}$-operators generate a new "calculus," which is original and many-sided and has both a "differential" side and an "integral" side;

(ii) the resurgent functions "algebrize" and thus simplify many analytical problems, especially in the study of singular ODEs;

(iii) they are more and more frequent in theoretical physics, where the frequent divergence is not a "curse" but rather a source of new insights.

We indicate some references to the topic under consideration: [1–7] (the reader is referred to the author's web site for a more complete list).

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Jean R. Ecalle
CNRS & Paris 11 Univ., Orsay, France
`jean.ecalle@math.u-psud.fr, homepage: http:/www.math.u-psud.fr/ẽcalle/`

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ASYMPTOTICALLY OPTIMAL ERRORS OF LATTICE CUBATURE FORMULAS

Ts. Zh. Yumova

Abstract. The norm of a periodic error is minimized by a functional analytical method of the cubature theory. The quality of the formulas constructed is improved for a large family of elements of an anisotropic space.

Keywords: lattice cubature formula, periodic error, anisotropic Sobolev space

Introduction. The statements and basic results of the theory of cubature formulas are due to Academician S. L. Sobolev [1] who proposed the functional analytical method for a family of integrands of a Hilbert space. His articles serve as fundamentals and sources of intensive development of the modern theory of partial differential equations, functional analysis, and numerical mathematics.

Multi-dimensional integrals are difficult to compute in view of bulky calculations, since at present there are no universal methods for optimization of cubature formulas on function spaces. Hence, the studies are realized from the standpoints of various research directions. One of these directions is the "functional analytical" approach connected with the study of error estimates in classes of summable functions and normed linear vector spaces of integrable functions.

The main results of optimization problems for cubature formulas over the anisotropic function spaces $W_p^{\overline{m}}(E_n)$ with different smoothness for different coordinate directions are exposed in the articles by Ts. B. Shoinzhurov [2, 3] and M. D. Ramazanov [4]. In particular, M. D. Ramazanov investigates cubature formulas on a nonweighted space of periodic functions with main period the unit cube. But with this definition of the norm of a function evokes certain difficulties with periodic extension of a function onto the unit cube. Ts. B. Shoinzhurov extends functions from their domains by "removing" constraints, which fact allows him to apply the Fourier transform to a function periodic on the whole space.

To construct lattice cubature formulas, asymptotically optimal with respect to integrable functions and depending on their differential properties, we explicitly present the main term of the norm of a periodic error with finite summability exponent.

Preliminaries. Assume that $h_k > 0$, $k = 1, \dots, n$, is the mesh width, x_k is a node of the formula, C_k is the coefficient of a formula, m_k is the smoothness of a function along the coordinate direction, $\overline{m} = (m_1, m_2, \dots, m_n)$, $m^* = n/\big(\sum_{k=1}^{n} m_k^{-1}\big)$, $\bar{h} = \operatorname{diag}(h_1, h_2, \dots, h_n)$ is the matrix of periods, $\Delta_{\bar{h}} = \{x \in E_n,\ 0 \le x_k < h_k,\ k = 1, 2, \dots, n\}$ is the fundamental parallelepiped with edge length h_k, $\Delta_{\bar{h}} = h^n = \det \bar{h} \neq 0$, and $\Delta = \{x \in E_n,\ 0 \le x_k < 1,\ k = 1, 2, \dots, n\}$ is the fundamental unit cube.

In the article we study cubature formulas on classes of periodic functions from an anisotropic space with the matrix $\bar{h}$ of periods whose generalized derivatives has different smoothness properties along different coordinate directions. The formulas constructed are used for a family of the integrands from the anisotropic space $W_p^{\overline{m}}(E_n)$ endowed with the natural norm

$$\|\varphi\|_{W_p^{\overline{m}}(E_n)} = \left[\int\limits_{E_n} \left(|\varphi(x)|^p + \sum_{k=1}^{n} |D^{m_k}\varphi(x)|^p \right) dx \right]^{1/p} < \infty.$$

The difference between the integral and an approximating combination of the values of the integrand is treated as the result of applying some generalized function that is defined by the cubature formula and is called an error.

The cubature sum approximating an integral from the functional standpoint is spanned by the Dirac delta functions $\delta(x)$, i.e.

$$\sum_{k=1}^{N} C_k \varphi(x^{(k)}) = \left(\sum_{k=1}^{N} C_k \delta(x - x^{(k)}), \varphi(x) \right).$$

The delta-functions can be applied only to a continuous test function and so we require the embedding of the basic space into the space of continuous functions $W_p^{\overline{m}}(E_n) \subset C(E_n)$, which is ensured by the embedding condition $p - \sum_{k=1}^{n} m_k^{-1} > 0$ (see [5]). This embedding is continuous, i.e., the error of a cubature formula is linear and bounded on $W_p^{\overline{m}}(E_n)$. The numerical majorant of its norm in the dual space $W_p^{\overline{m}^*}(E_n)$ allows us to obtain guaranteed estimates for proximity of the integral of this function and the cubature sum in question.

As is known [6], the differential properties of an anisotropic space are different in different directions; while the space is complete for $1 \le p \le \infty$, separable for $1 \le p < \infty$, and reflexive and uniformly convex for $1 < p < \infty$.

General representation of the error and the extremal function. A cubature formula is assumed to be of a better quality if its error has smaller norm. To find the norm of the error in the corresponding space, we involve the extremal function that is a generalized solution to some partial differential equations. The differential operator $L(D) = \sum_{k=0}^{n} (-1)^{m_k} D^{2m_k}$ occurring in such equation is generated by the form of the norm of a function in the basic space.

It is proven (see, for instance, [3]) that every linear functional $l(x)$ in $W_p^{\overline{m}^*}$ is representable as

$$\langle l, \varphi \rangle = \int\limits_{E_n} \sum_{k=0}^{n} (-1)^{m_k} D^{m_k} u * D^{m_k} \varphi \, dx,$$

where $\varphi \in W_p^{\overline{m}}$, $u \in W_{p'}^{\overline{m}}$ and

$$L(D)u = l(x), \quad l \in W_p^{\overline{m}^*}. \tag{1}$$

As is known [7], the fundamental solution $\varepsilon_{2\overline{m}}(x)$ to $L(D)$ is not unique, it is defined to within some summand $\varepsilon_{2\overline{m}}^0(x)$ that is a solution to the homogeneous equation $L(D)\varepsilon_{2\overline{m}}^0 = 0$. For $\varepsilon_{2\overline{m}}(x) \in W_p^{\overline{m}^*}(E_n)$ satisfying

$$L(D)\varepsilon_{2\overline{m}}(x) = \delta(x) \tag{2}$$

to be a fundamental solution to (1), it is necessary and sufficient that the Fourier transform of $\varepsilon_{2\overline{m}}(x)$ satisfy the equation $L(2\pi i\xi)F[\varepsilon_{2\overline{m}}(x)] = 1$, where

$$L(2\pi i\xi) = \sum_{k=1}^{n}(-1)^{m_k}(2\pi i\xi_k)^{2m_k} + 1 = 1 + \sum_{k=1}^{n}(2\pi\xi_k)^{2m_k}.$$

A fundamental solution $\varepsilon_{2\overline{m}}(x)$ to $L(D)$, called a kernel in what follows, is a function infinitely differentiable at $x \neq 0$, summable in E_n, and

$$\varepsilon_{2\overline{m}}(x) = F^{-1}\Bigg(\frac{1}{\sum\limits_{k=0}^{n}(-1)^{m_k}|2\pi\xi_k|^{2m_k}}\Bigg).$$

Owing to the special norm introduced by Ts. B. Shoinzhurov in [2] for which the corresponding differential operator was well studied and described in the literature (see, for instance, [5]), it is possible to apply the properties of its fundamental solution for finding extremal functions and norms of errors of cubature formulas. The general representations of the periodic error $\tilde{l}_0(h^{-1}x) = 1 - \Phi_0(h^{-1}x)$ is as follows:

$$\langle\tilde{l}_0(h^{-1}x), \varphi(x)\rangle_{\Delta_{\bar{h}}} = \int\limits_{\Delta_{\bar{h}}}\sum_{k=0}^{n}\big(D^{m_k}\varphi_0(h^{-1}x) + h_k^{m_k}C_k^{(0)}\big)D^{m_k}\varphi(x)\,dx,$$

and the corresponding extremal function

$$\varphi_0(h^{-1}x) = h^{m^*p'}\sum_{\gamma\neq 0}\frac{e^{2\pi ih^{-1}\gamma(x-y)}}{|2\pi\gamma|^{2m^*}}C_\gamma, \tag{3}$$

$$C_\gamma = \int\limits_{\Delta_{\bar{h}}}\sum_{k=0}^{n}(-1)^{m_k}\Bigg|C_k^{(0)}\sum_{\beta\neq 0}\frac{D^{m_k}e^{-2\pi ih^{-1}\beta y}}{1+\sum\limits_{j=1}^{n}(2\pi ih^{-1}\beta_j)^{2m_j}} + h_k^{m_k}C_k^{(0)}\Bigg|^{\frac{1}{p-1}}$$
$$\times\,\mathrm{sgn}\Bigg(C_k^{(0)}\sum_{\beta\neq 0}\frac{D^{m_k}e^{-2\pi ih^{-1}\beta y}}{1+\sum\limits_{j=1}^{n}(2\pi ih^{-1}\beta_j)^{2m_j}} + h_k^{m_k}C_k^{(0)}\Bigg)\,dy,$$

were obtained earlier in [8] under the condition $1 - p^{-1}\sum\limits_{k=1}^{n}m_k^{-1} > 0$. Here $C_k^{(0)}$ is a solution to the system of equations

$$\int\limits_{\Delta}\sum_{k=1}^{n}\Bigg|\frac{(-2\pi i\beta_k)^{m_k}e^{-2\pi i\beta_kx_k}}{\sum\limits_{j=0}^{n}(2\pi i\beta_j)^{2m_j}} + C_k^{(0)}\Bigg|^{\frac{1}{p-1}}\mathrm{sgn}\Bigg(\frac{(-2\pi i\beta_k)^{m_k}e^{-2\pi i\beta_kx_k}}{\sum\limits_{j=0}^{n}(2\pi i\beta_j)^{2m_j}} + C_k^{(0)}\Bigg)\,dx = 0. \tag{4}$$

An approach similar to that in [2] with the use of properties of the fundamental solution $\varepsilon_{2\overline{m}}(x)$ allows us to determine the norm of the periodic error of cubature formulas as follows:

$$\|\tilde{l}_0(h^{-1}x)\|_{\widetilde{W}_p^{\overline{m}*}(\Delta_{\bar{h}})}$$
$$= \Bigg(\int\limits_{\Delta_{\bar{h}}}\sum_{k=0}^{n}\Bigg|\sum_{\beta\neq 0}\frac{\big(2\pi ih_k^{-1}\beta_k\big)^{m_k}e^{-2\pi ih_k^{-1}\beta_kx_k}}{1+\sum\limits_{j=1}^{n}(2\pi ih^{-1}\beta_j)^{2m_j}} + h_k^{m_k}C_k^{(0)}\Bigg|^{p'}dx\Bigg)^{1/p'}. \tag{5}$$

Asymptotic representation for the norm of an optimal error. In the space of periodic functions we extract the main term of the norm of (5) which is independent of $\bar{h} = \sqrt{h_1^2 + h_2^2 + \cdots + h_n^2}$ when vanishes.

In view of the difficulty (see [4]) of characterization of an anisotropic space of functions periodic with the same basic period whose differential properties are different in different directions, we need to require that the norm of a function does not increase after subtracting the zero coefficient in the corresponding series. To resolve this problem, the authors of [8] obtain a criterion for asymptotic optimality of a cubature formula on coefficients, namely: the order of convergence must agree with the lattice mesh and smoothness of a functions in coordinate directions by the relations

$$h_1^{m_1} = h_2^{m_2} = \cdots = h_n^{m_n} = h^{m^*}. \tag{6}$$

It allows us to establish the order optimality on the class of lattice cubature formulas on an anisotropic space and to determine with the use of (6) the following dependence of the lattice mesh on smoothness of a function in a given direction:

$$h_k = h^{m^*/m_k}, \quad N_k = N^{m^*/m_k}, \quad k = 1, 2, \ldots, n.$$

Since the order of convergence must agree with the lattice mesh and smoothness of a function in coordinate directions by (6), we transform the integral on the right-hand side of (5) as follows:

$$J = \int\limits_{\Delta_{\bar{h}}} \sum_{k=0}^{n} \left| \sum_{\beta \neq 0} \frac{(-2\pi i h_k \beta_k)^{m_k} e^{-2\pi i h_k \beta_k x_k}}{\sum\limits_{j=0}^{n} (2\pi i h_k \beta_j)^{2m_j}} + h_k^{m_k} C_k^{(0)} \right|^{p'} dx$$

$$= h^{m^* p'} h^n \int\limits_{\Delta} \sum_{k=0}^{n} \left| \sum_{\beta \neq 0} \frac{(-2\pi i \beta_k)^{m_k} e^{-2\pi i \beta_k x_k}}{\sum\limits_{j=0}^{n} (2\pi i \beta_j)^{2m_j}} + C_k^{(0)} \right|^{p'} dx. \tag{7}$$

The equality

$$(-2\pi i \beta_k)^{m_k} e^{-2\pi i \beta x} = h^{m^*} e^{-2\pi i \beta x}$$

holds for the zero value of k and so (7) can be rewritten as

$$J = \int\limits_{\Delta} \left| \sum_{\beta \neq 0} \frac{h^{m^*} e^{-2\pi i \beta_k x_k}}{h^{2m^*} + \sum\limits_{j=1}^{n} (2\pi i \beta_j)^{2m_j}} \right|^{p'} dx + \sum_{k=1}^{n} \left| \sum_{\beta \neq 0} \frac{(-2\pi i \beta_k)^{m_k} e^{-2\pi i \beta_k x_k}}{h^{2m^*} + \sum\limits_{j=1}^{n} (2\pi i \beta_j)^{2m_j}} + C_k^{(0)} \right|^{p'} dx.$$

Theorem. *Assume that the conditions* (2) *and* (3) *hold in* $\widetilde{W}_p^{\overline{m}}(\Delta_{\bar{h}})$,

$$\tilde{l}_0(h^{-1}x) \in \widetilde{W}_p^{\overline{m}^*}(\Delta_{\bar{h}}), \quad 1 - p^{-1} \sum_{k=1}^{n} m_k^{-1} > 0, \quad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

h_k *is the lattice mesh, the norm of a functional* $\tilde{l}_0(h^{-1}x)$ *is defined by* (5), *with the parameters* $C_k^{(0)}$ *solutions to* (4). *Then the norm of an optimal periodic error as* $\bar{h} \to 0$ *has asymptotic representation*

$$\|\tilde{l}_0(h^{-1}x)\|_{\widetilde{W}_p^{\overline{m}}(\Delta_{\bar{h}})}$$

$$= h^{m^* + n + \frac{n}{p'}} \left(\int\limits_{\Delta} \sum_{k=1}^{n} \left| \sum_{\beta \neq 0} \frac{(-2\pi i \beta_k)^{m_k} e^{-2\pi i \beta_k x_k}}{h^{2m^*} + \sum\limits_{j=1}^{n} (2\pi i \beta_j)^{2m_j}} + C_k^{(0)} \right| \right)^{1/p'} (1 + O(h)). \tag{8}$$

Proof. In view of (6), taking $h_k^{-1}x_k = y_k$, $k = 1, 2, \dots, n$, in (5), we infer

$$\|\tilde{l}_0(h^{-1}x)\|_{\widetilde{W_p^m}(\Delta_{\bar{h}})}$$

$$= \Bigg(\int\limits_{\Delta_{\bar{h}}} \sum_{k=0}^{n} \Bigg| \sum_{\beta\neq 0} \frac{\big(-2\pi i h_k^{-1}\beta_k\big)^{m_k} e^{-2\pi i h_k^{-1}\beta_k x_k}}{\sum\limits_{j=0}^{n} \big(2\pi i h_j^{-1}\beta_j\big)^{2m_j}} + h_k^{m_k} C_k^{(0)} \Bigg|^{p'} dx\Bigg)^{1/p'}$$

$$= \left\langle \begin{matrix} y = h^{-1}x & dx = h^n dy \\ x = hy & \Delta_{\bar{h}} \to \Delta \end{matrix} \right\rangle$$

$$= h^n \Bigg(\int\limits_{\Delta} \sum_{k=0}^{n} \Bigg(\sum_{\beta\neq 0} \frac{(-2\pi i \beta_k)^{m_k} h_k^{-m_k} e^{-2\pi i h_k^{-1}\beta_k y_k}}{\sum\limits_{k=0}^{n} \big(2\pi i h_k^{-1}\beta_k\big)^{2m_k}} + h_k^{m_k} C_k^{(0)} \Bigg) dy\Bigg)^{1/p'}.$$

Transform the denominator as follows:

$$\sum_{k=0}^{n} \big(2\pi i h_k^{-1}\beta_k\big)^{2m_k} = \sum_{k=0}^{n} \big|2\pi h_k^{-1}\beta_k\big|^{2m_k} = \sum_{k=0}^{n} |2\pi\beta_k|^{2m_k} h_k^{-2m^*}$$

$$= h_k^{-2m^*} \sum_{k=0}^{n} |2\pi\beta_k|^{2m_k} h_k^{2m^*-2m_k}. \tag{9}$$

Next,

$$\sum_{k=0}^{n} |2\pi\beta_k|^{2m_k} h_k^{2m^*-2m_k} = h^{2m^*} + |2\pi\beta_1|^{2m_1} h_k^{2m^*-2m_1} + |2\pi\beta_2|^{2m_2} h_k^{2m^*-2m_2}$$

$$+ \dots + |2\pi\beta_{n-1}|^{2m_{n-1}} h_k^{2m^*-2m_{n-1}} + |2\pi\beta_n|^{2m^*} > |2\pi\beta_k|^{2m^*}. \tag{10}$$

Taking (9) and (10) into account, we conclude that

$$\|\tilde{l}_0(h^{-1}x)\| = h^n \Bigg(\int\limits_{\Delta} \sum_{k=0}^{n} \Bigg(\sum_{\beta\neq 0} \frac{(-2\pi i \beta_k)^{m_k} h_k^{-m_k} e^{-2\pi i \beta_k y_k}}{-h_k^{-2m^*} \sum\limits_{k=0}^{n} (2\pi i \beta_k)^{2m_k} h_k^{2m^*-2m_k}} + h_k^{m_k} C_k^{(0)} \Bigg) dy\Bigg)^{1/p'}$$

$$= h^n \Bigg(\int\limits_{\Delta} \sum_{k=0}^{n} \Bigg(\sum_{\beta\neq 0} \frac{(-2\pi i \beta_k)^{m_k} h_k^{2m^*-m_k} e^{-2\pi i \beta_k y_k}}{-\sum\limits_{k=0}^{n} (2\pi i \beta_k)^{2m_k} h_k^{2m^*-2m_k}} + h_k^{m_k} C_k^{(0)} \Bigg) dy\Bigg)^{1/p'}$$

$$= h^n \Bigg(\int\limits_{\Delta} \sum_{k=0}^{n} h_k^{(2m^*-m_k)p'} \Bigg(\sum_{\beta\neq 0} \frac{(-2\pi i \beta_k)^{m_k} e^{-2\pi i \beta_k y_k}}{-\sum\limits_{k=0}^{n} (2\pi i \beta_k)^{2m_k} h_k^{2m^*-2m_k}} + C_k^{(0)} \Bigg) dy\Bigg)^{1/p'}$$

$$\le K h^n \sum_{k=0}^{n} h_k^{(2m^*-m_k)p'} \Bigg(\int_{\Delta} \Bigg(\sum_{\beta\neq 0} \frac{(-2\pi i \beta_k)^{m_k} e^{-2\pi i \beta_k y_k}}{-\sum\limits_{k=0}^{n} (2\pi i \beta_k)^{2m_k} h_k^{2m^*-2m_k}} + C_k^{(0)} \Bigg) dy\Bigg)^{1/p'}. \tag{11}$$

Rewrite (11) as follows:

$$\|\tilde{l}_0(h^{-1}x)\|^{p'} \le K h^n \sum_{k=0}^{n} \int\limits_{\Delta} \sum_{k=0}^{n} \Bigg| \sum_{\beta\neq 0} \frac{(-2\pi i \beta_k)^{m_k} e^{-2\pi i \beta_k x_k}}{1 + \sum\limits_{k=1}^{n} (2\pi\beta_k)^{2m_k}} + C_k \Bigg|^{p'} dy h_k^{(2m^*-m_k)p'}$$

$$= Kh^n \int\limits_{\Delta_{\bar h}} \sum_{k=0}^{n} h^{2m^* p'} \Bigg| \sum_{\beta \neq 0} \frac{(-2\pi i \beta_0)^{m_0} e^{-2\pi i \beta_0 x_0}}{1 + \sum\limits_{k=1}^{n} (2\pi \beta_k)^{2m_k}} + C_0 \Bigg|^{p'} dy$$

$$+ Kh^n \int\limits_{\Delta_{\bar h}} \sum_{k=0}^{n} h^{(2m^* - m_1) p'} \Bigg| \sum_{\beta \neq 0} \frac{(-2\pi i \beta_1)^{m_1} e^{-2\pi i \beta_1 x_1}}{1 + \sum\limits_{k=1}^{n} (2\pi \beta_k)^{2m_k}} + C_1 \Bigg|^{p'} dy$$

$$+ \cdots + Kh^n \int\limits_{\Delta_{\bar h}} \sum_{k=0}^{n} h^{(2m^* - m_n) p'} \Bigg| \sum_{\beta \neq 0} \frac{(-2\pi i \beta_n)^{m_n} e^{-2\pi i \beta_n x_n}}{1 + \sum\limits_{k=1}^{n} (2\pi \beta_k)^{2m_k}} + C_n \Bigg|^{p'} dy$$

$$\leq Kh^n \int\limits_{\Delta} \sum_{k=0}^{n} h^{m^* p'} \Bigg| \sum_{\beta \neq 0} \frac{(-2\pi i \beta_k)^{m_k} e^{-2\pi i \beta_k x_k}}{1 + \sum\limits_{k=1}^{n} (2\pi \beta_k)^{2m_k}} + C_k \Bigg|^{p'} dy (1 + K_1 h^{p'}).$$

Since smoothness of functions in an anisotropic space is different in different directions with the lattice mesh h_k, we find that

$$\|\tilde l_0(h^{-1}x)\|^{p'} \leq Kh^n \int\limits_{\Delta} \sum_{k=0}^{n} h^{(m^*+n)p'} \Bigg| \sum_{\beta \neq 0} \frac{(-2\pi i \beta_k)^{m_k} e^{-2\pi i \beta_k x_k}}{1 + \sum\limits_{k=1}^{n} (2\pi \beta_k)^{2m_k}} + C_k \Bigg|^{p'} dy (1 + O(h^{p'})),$$

$$\|\tilde l_0(h^{-1}x)\| \leq K h^{(m^*+n)} h^{\frac{n}{p'}} \Bigg(\int\limits_{\Delta} \sum_{k=1}^{n} \Bigg| \sum_{\beta \neq 0} \frac{(-2\pi i \beta_k)^{m_k} e^{-2\pi i \beta x}}{\sum\limits_{j=1}^{n} (2\pi i \beta_j)^{2m_j}} + C_k^{(0)} \Bigg| \Bigg)^{1/p'} (1 + O(h)).$$

Thus, the norm (4) as $\bar h \to 0$ has asymptotic representation (8).

The theorem is proven.

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Ts. Zh. Yumova
East Siberia State University of Technology and Management
Ylan-Ude, Russia
syum@mail.ru

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THE INFLUENCE OF NONLINEAR EFFECTS ON THE SOCIETY STABILITY

E. K. Basaeva, E. S. Kamenetsky, and Z. Kh. Khosaeva

Abstract. We consider the interaction of the elite and workers in a society. Our qualitative analysis of the differential equations describing this interaction shows that a fold catastrophe can occur when the number of real stationary points changes. The form of the fold depends on the coefficient characterizing the intrinsic tendency to reduce or increase influence.

Keywords: dynamical model, stationary points, fold catastrophe

No adequate dynamical models are available to describe social systems on the microlevel, i.e., on the level of interactions between separate individua and/or small groups. At the same time, numerous attempts were made at applying dynamical models based on reasonable and realistic social hypotheses, i.e., information on the driving social forces, to analyze the interactions between large social groups (the macrolevel), see the survey [1]. In this case the problem arises of extracting the macroparameters essential for a particular study and write down the equations determining how the macroparameters change.

The nonlinear character of state social systems change is generally accepted presently. Nonlinear effects can lead to unstabilities of the system (see [2–6] for instance); thus, the appearance of social crises and revolutions is related to nonlinear effects. Some nonlinear model application to describing the interaction between the ruling group and a hostile social group striving for power exemplify in [5]. As the controlling parameter grows above a certain threshold value, that system of equations leads to a sharp jump in the hostility level between the groups; i.e., a fold catastrophe occurs. The controlling parameter is the level of initial discontent of the group striving for power, which is related to worsening economic situation, increasing inequality, the feeling of discrimination in ethnic, religious, language, and regional groups, as well as the introduction of political sanctions. There are no estimates in [5] for the coefficients of the model and the influence of these factors on the controlling parameter.

The model of collective actions proposed in [7], considers the influence of emotions and stereotypes on the people's actions in a crowd. This model describes the interaction between individuals in a group. We can apply the same approach to describe the interaction among several social groups. As an integral characteristic of the mental state of the group we take the social and political tension.

By *social and political tension* we understand the integral phenomenon reflecting the dissatisfaction of a large number of people with the system of social, economic, and political relations.

To describe tension in a social group, we use the differential equation that is obtained by simple transformations and translation to the limit from the finite difference equation of [7]:

$$\frac{dP}{dt} = \gamma[1 - \eta(1 - \beta)]U + \gamma[\eta((U + \beta(1 - 2U)) - 1]P,$$

where $P \in [0, 1]$ (with $P = 0$ corresponding to the total absence of tension, and $P = 1$ to the maximal possible tension), while U is the controlling parameter, γ is the intensity of the influence perception, $\eta \in [0, 1]$ is the internal tendency to reduce or increase the influence, and $\beta \in [-1, 1]$ is the perceptiveness to affects.

If the internal tendency to increase the influence is absent (i.e., $\eta = 0$) then

$$\frac{dP}{dt} = \gamma(U - P). \tag{1}$$

This equation can describe the dependence of tension in the group on the changing economic situation or another, less influential social group. Note that sharp worsening of the economic situation corresponds to $U \to 1$, while sharp improvement, to $U \to 0$.

When the intergroup influence is essential, for instance, when $\beta = 1$, we have

$$\frac{dP}{dt} = \gamma[(U - P) + \eta P(1 - U)]. \tag{2}$$

However, when tension in a social group is substantially determined by the influence of another group, it is expedient [8] to assume that the intensivity coefficient γ is not constant and write it as $\gamma = cU/(1-U)$. Here the constant $c \in [0, 4]$ determines the danger perception degree for this social group related to a conflict with the other group.

Consider the change of tension in a society consisting of the two groups: the ruling elite and the workers. Assume that the influence of changing economic situation and intergroup interaction is additive, while the influence of the workers on the elite is minor. Denote tension in the elite by P_1 and in the workers by P_2. Then tension in the elite consists of the two terms of the form (1): $\gamma_{э1}(U_1 - P_1)$ is the influence of economic factors, and $\gamma_{12}(P_2 - P_1)$ is the influence of the workers on the elite, where $U = P_2$. Thus,

$$\frac{dP_1}{dt} = \gamma_{э1}(U_1 - P_1) + \gamma_{12}(P_2 - P_1). \tag{3}$$

If the elite guarantees itself the maximally favorable economic situation, i.e., $U_1 = 0$; then

$$\frac{dP_1}{dt} = -\gamma_{э1}P_1 + \gamma_{12}(P_2 - P_1). \tag{4}$$

Similarly, tension in the workers is the sum of a term of the form (1) reflecting the influence of changes in economic situation on the workers and a term of the form (2) characterizing the influence of the elite on the workers. Since tension in the workers is considerably influenced by the elite, assume that $\gamma_{21} = c_2P_1/(1 - P_1)$. Therefore, we arrive at the following equation describing tension in the workers P_2:

$$\frac{dP_2}{dt} = \gamma_{э2}(U_2 - P_2) + c_2\frac{P_1}{1 - P_1}[(P_1 - P_2) + \eta_2P_2(1 - P_1)]. \tag{5}$$

The system of (4) and (5) coincides with the equations proposed in [5] in the case that "the cost of collective action is proportional to the squared tension in the ruling elite."

Let us find the stationary points of the resulting system of equations. From (4) we obtain the relation between the stationary value of tension in the elite, P_1^*, and the workers, P_2^*:

$$P_1^* = \frac{\gamma_{12}}{\gamma_{э1} + \gamma_{12}} P_2^*.$$

Inserting this expression into (5), we obtain

$$\eta_2 c_2 \left(\frac{\gamma_{12}}{\gamma_{э1} + \gamma_{12}}\right)^2 (P_2^*)^3 + \left(\gamma_{э2} + \frac{\gamma_{12}\gamma_{э2}}{\gamma_{э1} + \gamma_{12}} U_2\right) P_2^* - \gamma_{э2} U_2$$
$$- \left[\frac{\gamma_{12}\gamma_{э2}}{\gamma_{э1} + \gamma_{12}} + c_2 \frac{\gamma_{12}}{\gamma_{э1} + \gamma_{12}} \left(\frac{\gamma_{12}}{\gamma_{э1} + \gamma_{12}} - 1\right) + \eta_2 c_2 \frac{\gamma_{12}}{\gamma_{э1} + \gamma_{12}}\right] (P_2^*)^2 = 0.$$

Depending on U_2, for $\gamma_{э1} = 0.1$, $\gamma_{э2} = 0.1$, $\eta_2 = 0.4$, $c_1 = 0.3$, and $\gamma_{12} = 1$, the resulting cubic equation for P_2^* can have one or three real solutions (Fig. 1). The lower and upper branches of the solution correspond to stable stationary points, while the middle branch is unstable. This means that in transition from three stationary points to one tension in the workers may change sharply, i.e., a fold catastrophe occurs for a certain critical value of the controlling parameter. For greater values of the controlling parameter there is one stationary point with a large value of tension in the workers.

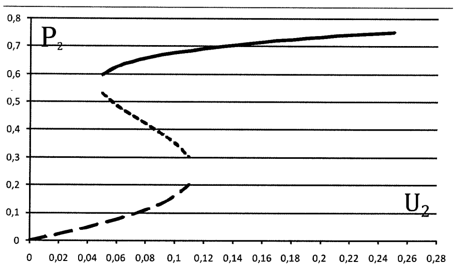

Fig. 1. Change of tension in the workers as the economic situation changes

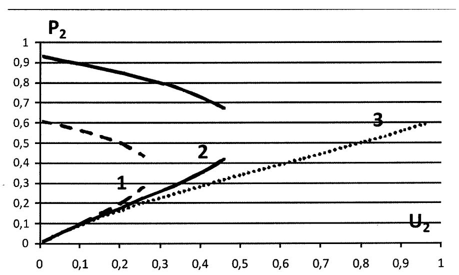

Fig. 2. Change of tension in the workers as the economic situation changes (curve 1: $\eta_2 = 0.6$, curve 2: $\eta_2 = 0.4$, curve 3: $\eta_2 = 0.2$)

The critical value $U_2 \approx 0.11$ of the controlling parameter obtained for the specified values of constants corresponds to a growing economy, and therefore it is unreal. Moreover, we have not managed to increase substantially this critical value for any values of constants. This means that the model (4), (5) is unsatisfactory.

To improve the model, suppose that the influence of workers on the elite is described by an expression similar to that for the influence of the elite on the workers, but with different constants: $\gamma_{12} = c_1 P_2/(1 - P_2)$ and $\eta_1 = 0$. The first equation of the system becomes

$$\frac{dP_1}{dt} = -\gamma_{э1} P_1 + c_1 \frac{P_2}{1 - P_2}(P_2 - P_1). \tag{4'}$$

Some algebraic equation of the six degree is obtained for the stationary points of the system (4′), (5).

The constants in ($4'$) and (5) are determined by the developed intergroup relations and mentality and change rather slowly (the characteristic time is of the order of decades). The parameter η_2, the internal tendency to reduce or increase the influence, is an exception; under the influence of information it can change quickly. Therefore, it is of interest to study the behavior of ($4'$), (5) for various values of η_2.

Using the model ($4'$), (5), we ran a series of computational experiments. For $\gamma_{э1} = 0.1$, $\gamma_{э2} = 0.1$, $c_1 = 0.3$, and $c_2 = 0.8$, depending on the value of η_2, one or two branches of the solution to the algebraic equation for the stationary points lie (Fig. 2) in the domain $P_2 \in (0,\ 1)$.

The lower branch is stable, whereas the upper branch is unstable. As η_2 increases, so does the distance between the branches, and the upper branch leaves the region of interest.

This behavior of the system corresponds to the understanding that society may destabilize under the influence of a sharply worsening economic situation (Fig. 2, curves 1 and 2). Observe that for sufficiently small values of η_2 destabilization never occurs at all (Fig. 2, curve 3). The reason for this is the appearance of feedback for large η_2. Increasing tension in the workers leads to increasing tension in the elite, which causes an additional increase of tension in the workers.

Note that, as usual, catastrophe theory implicitly assumes an instantaneous transition from one state into another, i.e., a jump from one equilibrium point to another or to the boundary of the domain. Actually, for the social system under consideration this process takes some time. Hence, if during this time the controlling parameter will return to the original value (which corresponds to improving the economic situation), while the state of the system still lies in the basin of attraction of the stable stationary point, then the system returns into that stable state.

Previously it was thought [9] that "the drop in living standards by one order of magnitude means a defeat, a revolutionary situation, or a change of the regime." However, already it is noted in [10] that "the events of recent years in a series of states, which we witnessed, show that this is false." Our results suggest explanation for the ambiguity of the influence of economic factors on the destabilization of society; i.e., apart from economic factors, the parameter η_2 also affects destabilization, and it apparently depends on the share of young people in the society [11], the presence of well-off and/or well-educated people devoid the possibilities of influencing the society management processes [12], and the information influence often external character [13].

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E. K. Basaeva
Southern Mathematical Institute
North Ossetia State University, Vladikavkaz, Russia
helen@smath.ru

E. S. Kamenetsky
Southern Mathematical Institute, Vladikavkaz, Russia
esk@smath.ru

Z. Kh. Khosaeva
Vladikavkaz Science Center of RAS, Vladikavkaz, Russia
hzaiac83@mail.ru

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3D MODELING OF SEISMIC WAVE FIELDS IN A MEDIUM SPECIFIC TO VOLCANIC STRUCTURES

B. M. Glinskiĭ, V. N. Martynov, and A. F. Sapetina

Abstract. The problem of predicting the catastrophic events that could be caused by an impending outbreak of volcanic activity is urgent. To solve this problem, we must conduct comprehensive and unbiased investigations including the numerical modeling of the processes occurring at the surface and inside a volcanic structure. This should be done in order to create a vibroseismic monitoring system. We have developed the parallel 2D and 3D algorithms and the programs for simulating the elastic wave propagation in media of complex subsurface 3D geometries (the 2D models are cross-sections of the original 3D model by planes at various angles). We use explicit finite difference schemes for staggered grids and the CFS-PML method of absorbing boundaries. The proposed numerical method and its parallel implementation efficiently use the modern supercomputer architecture that is based on graphic accelerators. For creating a possible system of vibroseismic monitoring of the strata-volcano Elbrus we have carried out a series of 3D calculations aimed at studying the structure of a wave field, generated by the geometry of internal boundaries of a certain model, and at refining its kinematic and dynamic characteristics.

Keywords: monitoring, 3D simulation, elastic waves, finite difference schemes, hybrid cluster, GPU

1. Introduction

In the present state-of-the-art, the software tools for carrying out the numerical modeling Are becoming increasingly dependent on supercomputer architecture, taming r not only on the choice of algorithms for solving the problems posed on the chosen architecture, but also on the choice of the statement as it is. Many algorithms work faster on appropriate types of architecture, and we often need to adapt a chosen algorithm to the available architecture, which can yield a large performance gain when solving the problem. Thus, the time needed for a simulation is decreased. This paper is aimed at designing the software tools for carrying out the numerical modeling of seismic wave fields in the 3D media characteristic of volcanic structures.

Volcanoes, including inactive ones, pose potential threats of sudden strong catastrophic events. The ability to predict an imminent eruption would enable us to save human lives and their property. Hence, it is necessary to monitor the state of volcanoes, to trace changes in parameters of their interior, and to make correct conclusions. The latter requires a preliminary thorough and comprehensive study of the processes occurring inside volcanoes and at their surfaces.

Active vibroseismic monitoring is one of the appropriate tools. The information it brings is difficult to interpret and requires preliminary and concurrent numerical modeling of the processes inside a geophysical object in question with allowance for its specific structure. Usually the shape of an object is complex that makes difficult to place An observational station for solving the inverse geophysics problem. Therefore, we have to solve the direct problem by varying the parameters of a simulated medium so that the results of the numerical and field experiments coincide.

In addition, this approach enables us to consider different possible types of the composition of magma volcanoes and their various states in order to distinguish the main effects appearing in the data recorded by the surface observational system at the surface of a volcano. The approach proposed facilitates the interpretation in the immediate monitoring.

Thus, when carrying out the vibroseismic monitoring we are facing the problem of the large-scale simulation of the processes occurring in volcanoes of various compositions.

Solving this problem requires the use of the supercomputer technologies of the latest design.

2. Statement of the Problem and the Method of Solution

To simulate seismic waves in elastic inhomogeneous media, we solve the complete system of elasticity equations with appropriate initial and boundary conditions written down in terms of the displacement velocity vector $\vec{u} = (U, V, W)^T$ and the stress tensor $\vec{\sigma} = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz})^T$.

As the domain of simulation we take an isotropic 3D-inhomogeneous elastic medium of complex subsurface geometry which is a parallele piped one of whose sides is a free surface (the plane $z = 0$).

The constitutive equations can be expressed in the vector form as

$$\rho \frac{\partial \vec{u}}{\partial t} = [A]\vec{\sigma} + \vec{F}(t, x, y, z), \quad \frac{\partial \vec{\sigma}}{\partial t} = [B]\vec{u},$$

$$A = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix},$$

$$B = \begin{bmatrix} (\lambda + 2\mu)\frac{\partial}{\partial x} & \lambda\frac{\partial}{\partial y} & \lambda\frac{\partial}{\partial z} \\ \lambda\frac{\partial}{\partial x} & (\lambda + 2\mu)\frac{\partial}{\partial y} & \lambda\frac{\partial}{\partial z} \\ \lambda\frac{\partial}{\partial x} & \lambda\frac{\partial}{\partial y} & (\lambda + 2\mu)\frac{\partial}{\partial z} \\ \mu\frac{\partial}{\partial y} & \mu\frac{\partial}{\partial x} & 0 \\ \mu\frac{\partial}{\partial z} & 0 & \mu\frac{\partial}{\partial x} \\ 0 & \mu\frac{\partial}{\partial z} & \mu\frac{\partial}{\partial y} \end{bmatrix}, \tag{1}$$

where t is the time, $\rho(x, y, z)$ is the density, while $\lambda(x, y, z)$ and $\mu(x, y, z)$ are the Lamé coefficients. The initial conditions are the following:

$$\sigma_{xz}|_{t=0} = 0,\ \sigma_{yz}|_{t=0} = 0,\ \sigma_{xy}|_{t=0} = 0,\ \sigma_{xx}|_{t=0} = 0,\ \sigma_{yy}|_{t=0} = 0,\ \sigma_{zz}|_{t=0} = 0, \tag{2}$$

$$U|_{t=0} = 0, \quad V|_{t=0} = 0, \quad W|_{t=0} = 0,$$

and the boundary conditions at the free surface are:

$$\sigma_{xz}|_{z=0} = 0, \quad \sigma_{yz}|_{z=0} = 0, \quad \sigma_{zz}|_{z=0} = 0. \tag{3}$$

To numerically solve equations (1)-(3)

we apply the well-known Verrier finite difference method [1–3]. The calculation of its difference coefficients uses integral conservation laws and The method is of second order of approximation with respect to time and space [1]. In this paper we consider only uniform grids.

Let us present as an example a few finite difference equations of the scheme used.

$$\frac{\rho_{i,j,k}+\rho_{i-1,j,k}}{2}\frac{u^{n+1}_{i-\frac{1}{2},j,k}-u^{n}_{i-\frac{1}{2},j,k}}{\tau}=\frac{\sigma^{n+\frac{1}{2}}_{xxi,j,k}-\sigma^{n+\frac{1}{2}}_{xxi-1,j,k}}{\Delta x}$$

$$+\frac{\sigma^{n+\frac{1}{2}}_{xyi-\frac{1}{2},j+\frac{1}{2},k}-\sigma^{n+\frac{1}{2}}_{xyi-\frac{1}{2},j-\frac{1}{2},k}}{\Delta y}+\frac{\sigma^{n+\frac{1}{2}}_{xzi-\frac{1}{2},j,k+\frac{1}{2}}-\sigma^{n+\frac{1}{2}}_{xzi-\frac{1}{2},j,k-\frac{1}{2}}}{\Delta z}+f^{n}_{xi,j,k},$$

$$\frac{\sigma^{n+\frac{1}{2}}_{xzi-\frac{1}{2},j,k-\frac{1}{2}}-\sigma^{n-\frac{1}{2}}_{xzi-\frac{1}{2},j,k-\frac{1}{2}}}{\tau}$$

$$=\mu 1_{i-\frac{1}{2},j,k-\frac{1}{2}}\left(\frac{u^{n}_{i-\frac{1}{2},j,k}-u^{n}_{i-\frac{1}{2},j,k-1}}{\Delta z}+\frac{w^{n}_{i,j,k-\frac{1}{2}}-w^{n}_{i-1,j,k-\frac{1}{2}}}{\Delta x}\right),$$

where

$$\mu 1_{i-\frac{1}{2},j,k-\frac{1}{2}}=\Big(\frac{1}{4}\Big(\frac{1}{\mu_{i,j,k}}+\frac{1}{\mu_{i-1,j,k}}+\frac{1}{\mu_{i,j,k-1}}+\frac{1}{\mu_{i-1,j,k-1}}\Big)\Big)^{-1}.$$

The boundaries of the simulated domain cause false reflections inside it. To absorb them, we use the auxiliary method CFS-PML [4–6], which has some advantages over the classical PML method. It yields a better qualitative picture of the wave field for this problem, is simpler to implement, and is efficient from the standpoint of doing the calculation.l экономичен

To apply this method, each of the boundaries of the parallelepiped,i/e/ the domain to be simulated, is bounded by the absorbing layer except for the free surface on its upper side.

We calculate the wave field interior using the original finite difference equations, but when a wave arrives in the absorption zone, the calculation is carried out by other formulas with damping parameters which describe the approach to creating absorbing boundaries. To choose the values of damping parameters for the calculation in appropriate absorbing layers, is made based on the results obtained in [5].

3. Program Package for Carrying out Simulations

In this paper we propose an approach to simulating the seismic wave propagation

in the media specific to volcanic structures. In this connection we need to develop a enabling us to construct a grid model of a medium of complex subsurface geometry and to carry out necessary calculations. In this case depending on our intent and available resources, we propose to perform either 3D modeling in the entire domain of interest, or 2D modeling in the cros s-sections of the original domain containing the most interesting features of the wave field, or to combine both approaches.

Therefore, the proposed program package must include the following parts:

—a program for constructing grid models of media of complex subsurface geometries with inclusions characteristic of magma volcanoes;

—a program for the numerical modeling of the elastic waves propagation in 3D inhomogeneous elastic media with a curvilinear free surface;

—a program for the numerical modeling of elastic waves propagation in 3D inhomogeneous elastic media with a rectilinear free surface;

—a program for the numerical modeling of the elastic waves propagation in 2D-inhomogeneous elastic media with a curvilinear free surface for a prescribed section of the 3D model under consideration;

—a program for simulating the elastic waves propagation in 2D-inhomogeneous elastic media with a rectilinear free surface for a prescribed section of the 3D model under consideration.

By now the proposed program package has been partially implemented. This enables us to handle the case of a rectilinear free surface. The programs were developed with allowance for specific features of the architecture of the hybrid cluster HKC-30T+GPU (the Siberian Supercomputing Center (http://www2.sscc.ru), which consists of 40 computer nodes equipped with NVIDIA Tesla M2090 graphics cards on the Fermi architecture. Its peak performance is 85 teraflops.

The efficient use of the hybrid architecture requires adapting and optimizing the simulation algorithms that are based on the knowledge of the architecture of the cluster, its components, and appropriate program facilities. A detailed description of the developed program tools as well as a parallel implementation, adaptation, and optimization of the algorithms is given in [7, 8]. Let us recall a few key points.

The developed program package includes a constructor of media models enabling us to design complicated elastic media models based on the idea of Z order [2, 3]. In the parallel implementation data are immediately constructed at the computer nodes to be used in the subsequent calculation.

Moreover, we have created a parallel program for the numerical modeling of wave propagation in three-dimensional inhomogeneous elastic media with a rectilinear free surface. This program implements the above-mentioned Verrier and CFS-PML methods adapted to the hybrid cluster architecture.

To parallelize the problem in question, we decompose the domain into layers along one of the coordinate axes. Each layer is calculated at a separate node, where, in turn, it is sub-divided into sub-layers along the other coordinate axis according to the number of graphics accelerators at a node. In such implementation, each graphics card calculates its own grid domain inside the sub-layer at each timel step independent of others, except for points at the interface between two adjacent domains. These points are common to each of domains, and, to continue the calculation, it is necessary to exchange information about the required values among the graphics cards at the node and between the adjacent nodes in the directions of different coordinate axes. The exchanges are made using the MPI technology (Message Passing Interface) and graphics accelerators are controlled using the CUDA technology (Compute Unified Device Architecture). In this case, a parallel part of the code is carried out on each graphics card as a large number of threads. This hybrid approach ensures a high degree of parallelization.

The developed program package also includes a program for simulating the seismic wave propagation in a two-dimensional elastic medium. It has been developed based on the program for the three-dimensional case. To solve the 2D problem, we use the same difference method on staggered grids and the auxiliary CFS-PML method in the two-dimensional version.

The 2D calculation takes much less computer resources, including the memory for storing all necessary data. Therefore, the implementation in question employs only one computer node of the hybrid cluster with three graphics accelerators. As a result, the developed program enables us to complete the calculation in an acceptable time with a lesser number of resources as compared to the 3D version, which requires for calculation of solution almost all resources of the hybrid cluster and more time for calculation.

For example, constructing a model and calculating the wave field at a node with three graphics accelerators for the corresponding grids with respect to time and space and (6000×9000 nodes with respect to space and 25000 time steps) takes only 12 minutes.

4. Scalability of the Developed Software

To analyze the performance of the developed software, we study its strong and weak scalabilities, simulate the program execution on a large number of cores, and compare its running time on the hybrid cluster with a similar time for the program execution on a cluster with the classical MPP-architecture.

By the strong scalability we understand a decrease in the running time of one step of the same problem when more graphics cores per one graphics card are used. Studying the strong scalability

we can understand how efficiently the algorithms use the architecture of the graphics card as it is. By the weak scalability we understand the preservation of the calculation time of one step of the same volume of the problem when the number of graphics cards increases.

Figs. 1 and 2 present the results of investigations obtained with the program for the 3D simulation. It is clear from the graph in Fig. 1 that the problem is well-suited to the graphics card architecture: there is roughly 40-fold acceleration when all GPU cores are used in comparison with one GPU core.

It is clear from the graph in Fig. 2 that the efficiency of about 92% is attained

when increasing the number of graphics cards up to 30. Based on the collected data about the calculation time spent on the required components of the stress tensor and the displacement velocity vector, as well as the time for exchanging data among the computer nodes and graphics accelerators for the implemented software, we have employed the simulation of the algorithm of the numerical modeling of the seismic wave propagation in an elastic medium using the distributed agent-oriented system AGNES

developed at the Institute of Computational Mathematics and Mathematical Geophysics.

(The AGNES package (AGent NEtwork Simulator relies on Java Agent Development Framework (JADE), which is a powerful tool for creating multi-agent systems in Java consisting of three parts: an agent execution environment; a library of the basic classes required for developing agent systems; a collection of utilities enabling us to monitor and to administer the multi-agent system. To simulate large calculations, it is important that JADE is a FIPA-compatible distributed agent platform capable of using one or several computers (network nodes), on each one there should be only one virtual Java machine [9]. This system enables us to map a numerical algorithm onto a hypothetical supercomputer, to study its behavior, and to adjust the computational method. Examples when this approach is applied are presented in [10].

Fig. 3 presents the simulation results. The beginning of the simulation process

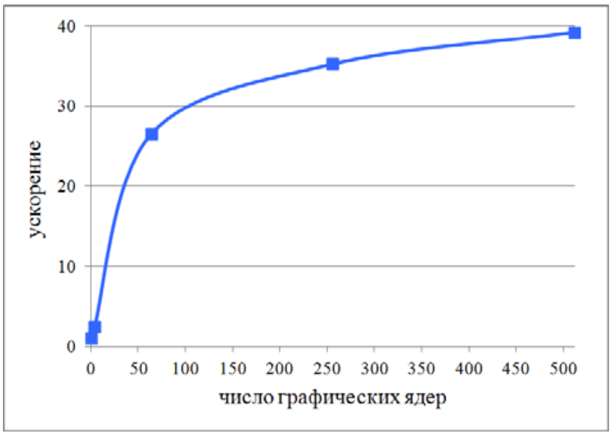


Fig. 1. Strong scalability graph

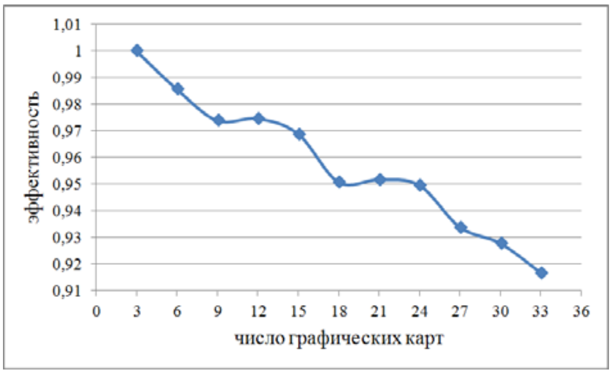


Fig. 2. Weak scalability graph

is compared with the real data of weak scalability. The simulation shows that as the number of computer cores increases up to one million, the efficiency of the proposed program is about 75%. This indicates to the fact that the developed software enjoys good scalability, which makes it suitable for large-scale supercomputer calculations.

We have compared the time spent on the numerical modeling of the 3D model with the use of nodes with GPU to the time needed for the calculation on the classical cluster with CPU. To this end, we use the time of the full-scale calculation [2] for the spatial grid of 1677x1059x971 size and $1677 \times 1059 \times 971$ and 10313 time steps. The calculation has been done on 20 computer blade- servers HP ProLiant BL2x220c G5, which are part of the HKC-30T supercomputer at the Institute of Computational Mathematics and Mathematical Geophysics. The time needed for the calculation was 31 h 15 m 17 s (160 CPU cores were involved).

For the same grid the time at 15 nodes of the hybrid cluster using the developed program package was 2 h 56 m (15360 GPU cores were involved); the resulting acceleration was by the factor of 10,66.

Thus, the developed software, adapted to modern hybrid computer architecture with graphics accelerators, used at many computer centers in Russia and abroad, demonstrates a high performance.

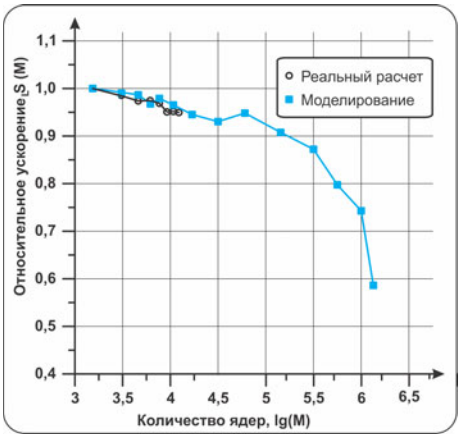


Fig. 3. The dependence of the relative acceleration $SL(M)$ on the general number M of simulated GPU cores (the horizontal axis is logarithmic)

5. Numerical Experiments

Based on the data of [11–15], as well as on data of other published works we have constructed a geophysical model of the Elbrus stratovolcano. The volcanic structure lies on a granite block +I (Fig. 4); effusive rocks build up the volcanic cone +II; below the zero mark we can distinguish eight layers (Table 1). Let us define the upper magma chamber as an ellipsoid with horizontal and vertical axes of 9 and 6 km ($\rho = 2,1\mathrm{g/cm}^3$, $V_p = 2,2$ km/s); the diameter of the former channel being 130 m. Let us define the parent magma chamber as an ellipsoid with horizontal and vertical axes of 24 and 13 km ($\rho = 1,8$ g/s^3, $V_p = 1,9$ km/s), and the diameter of the presumed feeding channel being 250 m. The middle channel is a cylinder of 160 m in diameter. Thus, as an approximate model of the Elbrus volcano we can take a multi-layer medium with inclusions as ellipsoids, cylinders with parameters listed in Table 1. A detailed description of the geophysical model can be found in [8].

Table 1. Parameters for the geophysical model of the Elbrus volcano

	V_p, km/s	V_s, km/s	ρ, g/cm^3
Layer +II	2,85	1,65	2,4
Layer +I	3,1	1,79	2,66
Layer I	3,2	1,82	2,7
Layer II	5,9	3,42	2,85
Layer III	6,22	3,59	2,62
Layer IV	5,82	3,37	2,7
Layer V	5,97	3,45	2,75
Layer VI	6,43	3,72	2,78
Layer VII	6,95	4,03	2,81
Layer VIII	8,1	4,68	2,85

The constructed model was taken as the basis for subsequent computer experi-

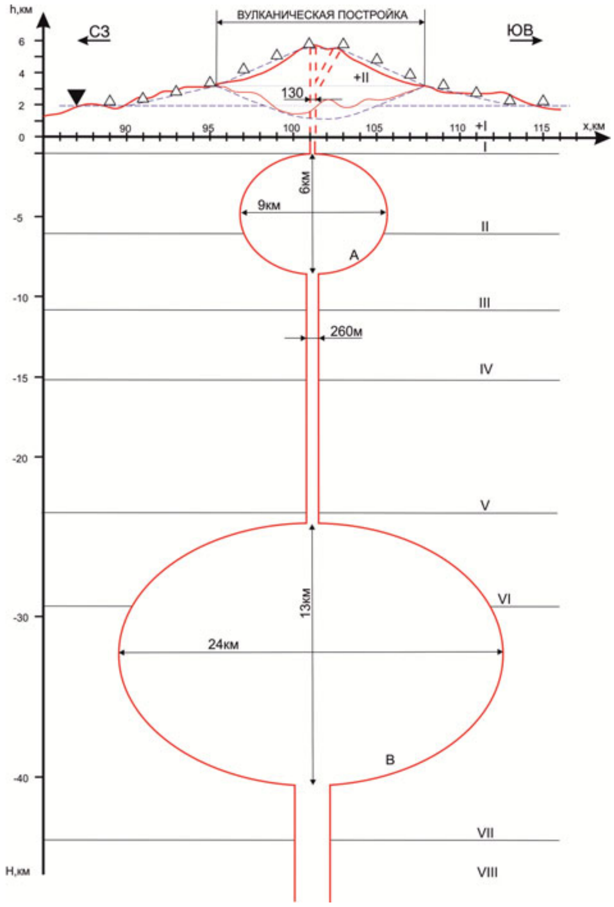


Fig. 4. Fig.4 The geophysical model of the Elbrus volcano and the scheme of vibroseismic monitoring

ments, whose results are presented below, to illustrate the efficiency of the developed program package as well as to demonstrate various possibilities of applying the proposed technology of supercomputer modeling of seismic wave propagation in the media typical of volcanic structures.

All the calculations were done at 11 nodes of the clusterHKC-30T+GPU equipped with graphics cards. The simulation with 12,000 time steps requires, on average, 1.5 hours.

The results of the 3D numerical modeling contain a large bulk of information. For this reason, to represent and to analyze the results of numerical modeling we use theoretical seismograms and snapshots of different sections of the 3D wave field by the planes passing through the lines of a selected observational system.

The excitation system for all calculations consists of a point source of the pressure center type with the dominant frequency 8 Hz which is near the free surface on the left side of the calculation domain in one plane with the symmetry axis of the magma channels and chambers.

For the beginning of calculation we took as a medium a fragment of the original approximate model of the Elbrus stratovolcano, which includes only the upper magma chamber of a modified shape and the adjacent channels lying in 5-layer medium. We used the parameters of the medium given in Table 1. The calculation was aimed at illustrating the potentialities of the constructor of grid models of the medium, with the aid of which we can construct a medium with inclusions of a complicated shape, for example, the intersection of several objects of different nature at different angles. In this case the magma chamber has the shape of two intersecting ellipsoids.

Fig. 5 shows the results of the calculation as snapshots of the wave field in the plane XZ passing through the point source and the symmetry axis of the upper channel. The snapshots were visualized using the program Aspis developed by the Sibneftegeofizika company. In the snapshots the boundaries of domains between parts with different parameters of the simulated medium are marked.

It is clear from Fig. 5 that the wave field has a complicated structure and significantly depends on the geometry, size, and properties of inclusions. To make a geophysical interpretation of the theoretical seismograms resulting from the calculation of such a complicated medium a large series of computer experiments for the 3D model of the medium as well as of its characteristic 2D sections is needed.

The results of the 3D and the 2D calculations for a fragment of the proposed approximate model of the Elbrus stratovolcano are compared in [8]. The presented snapshots illustrate our assumption that The 2D modeling can be used to construct 3D models of a medium. The calculation can be simultaneously done for a 3D model and several 2D sections to collect necessary information about peculiar features of the wave field structure for the chosen geophysical model, the observational system, and the location of a source. This approach considerably speeds up the process of studying the features of a wave field when modeling of vibroseismic sounding of a volcanic structure and enables us to do fewer calculations for the three-dimensional problem, requiring more computer costs.

Developing this software suite, we aimed at studying the processes inside magma volcanoes and on their surface in order to monitor subsequent eruptions. The following calculations could be the first step to attain these ends.

As a simulated medium we take a fragment of the proposed approximate model of the Elbrus stratovolcano, which includes only the upper magma chamber of the ellipsoidal shape and the adjacent feeding channel lying in 5-layer medium. The difference between these media is that in one case the upper channel is filled with magma (an eruption), while in the other case it merges with the surrounding layers . The parameters of the medium are as described above.

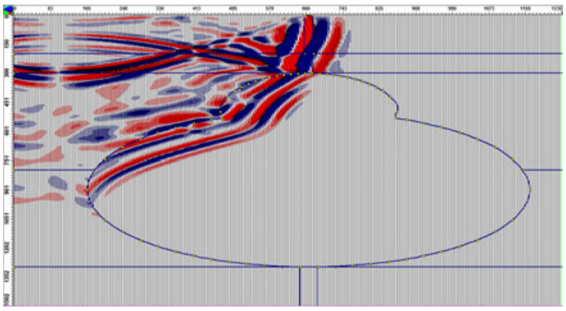
Figures 6, 7 present, respectively, the results of calculations as snapshots and theoretical seismograms of the wave field in the

plane XZ passing through the source and the symmetry axis of the upper channel. In these figures the difference in the results obtained is clearly seen. A thorough study of theoretical seismograms based on the snapshots of the wave field and additional calculations can reveal the features to be used for the monitoring of eruptions.

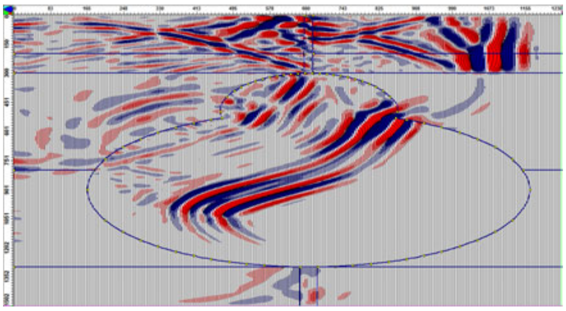
Interpreting the data obtained in the course of the numerical modeling for complex subsurface geometries with different inclusions is a difficult task. Nevertheless, the carried out experiments show that the numerical modeling can give significant information for conducting experiments and interpreting the results of observations obtained in the process of vibroseismic monitoring.

Fig. 7 presents theoretical seismograms which clearly show the differences arising in the general picture of media models.

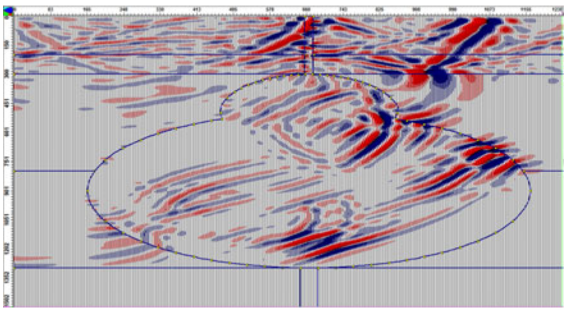
A careful study of theoretical seismograms based on these snapshots of the



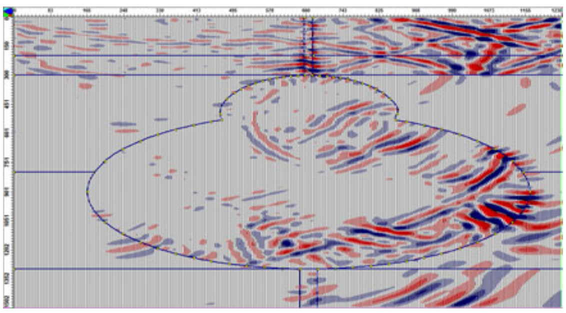
$T = 1.15$ s



$T = 1.84$ s

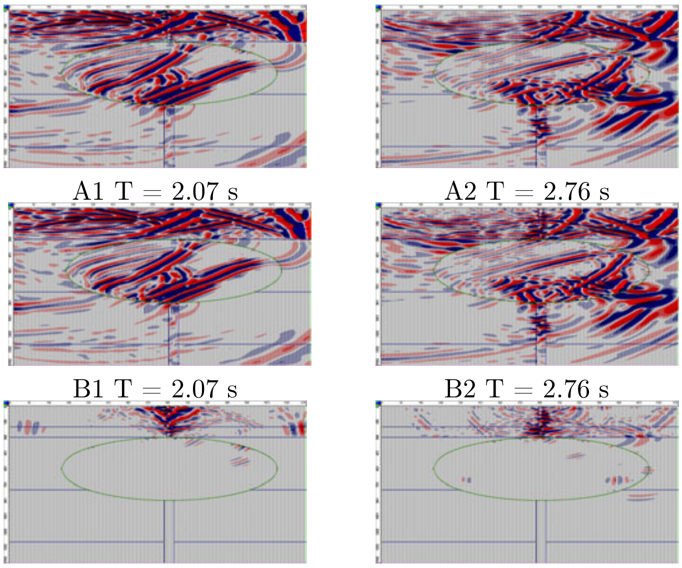


$T = 2.53$ s



$T = 3.22$ s

Fig. 5. Snapshots of cross-sections of 3D wave field
at various moments of time for a 5-layer medium
with the inclusion of two intersecting ellipsoids (component U, the XZ plane)

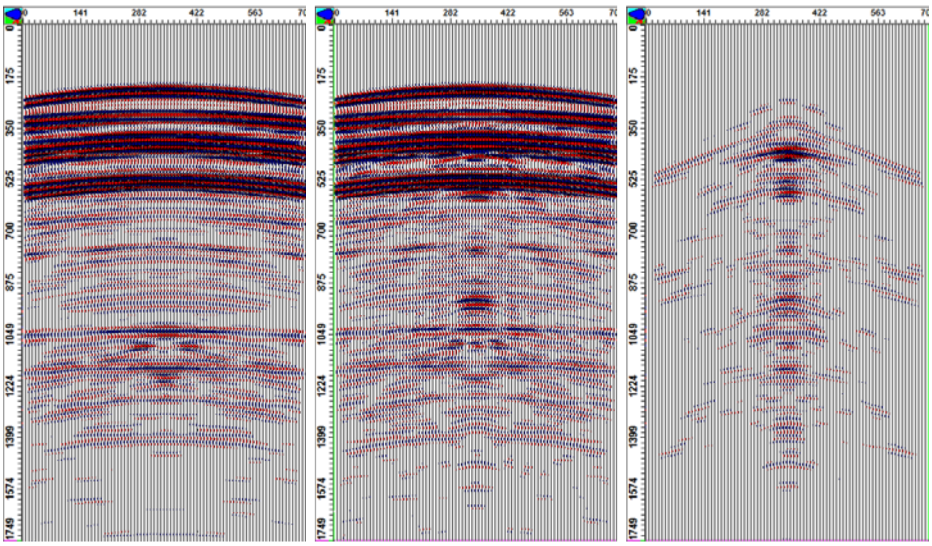


A1 T = 2.07 s A2 T = 2.76 s

B1 T = 2.07 s B2 T = 2.76 s

C1 T = 2.07 s C2 T = 2.76 s

Fig. 6. Snapshots of cross-sections of the 3D wave field
at different instants of time (component U, the XZ plane):
A—without upper channel
B—with the upper channel filled with magma (eruption),
C—the difference between seismograms A and B



A B C

Fig. 7. Theoretical seismograms for the components of U:
A—without upper channel
B—with the upper channel filled with magma (eruption),
C—the difference between seismograms A and B

wave field and additional calculations could be useful in the monitoring of volcanic structures.

6. Conclusion

This paper proposes a new technology for solving the problem of the elastic waves propagation in media of complex subsurface geometries with the use of different numerical methods for studying inhomogeneous media of different dimensions associated with the same model and aimed at the hybrid supercomputer architecture with graphics accelerators.

We have developed a bundle a parallel programs package with allowance for a chosen architecture to implement the simulation of the elastic wave propagation in 2D and 3D models of the media of complex subsurface geometries specific to volcanic structures. We have studied the time needed for the calculation of the model in question, including the simulation of the program execution on a large number of cores. The created software demonstrates high performance and can efficiently use the resources of large computer systems. We have carried out a series of experiments for simplified models of the Elbrus volcano. The conducted experiments have shown a high efficiency of the developed software and various possibilities for its applications in the subsequent research. The use of the proposed methodology makes possible to identify a model of the volcanic structure which would be adequate in its kinematic and dynamic characteristics to the results of experimental observations. Further we assume to develop the proposed technology of calculations for models with a curvilinear free surface. , and properties of inclusions. It is a difficult problem to interpret the data obtained in the simulations for media of compound structure with various inclusions required by an approximation to a real model of the medium. Nevertheless, the presented simulations show that modeling can yield important information for organizing field work and interpreting the results of vibroseismic monitoring. The particular features which can be gleaned from the results can be useful in vibroseismic monitoring of volcanic structures. Making a series of similar simulations enables us to choose a model of volcanic structure adequate in kinematical and dynamic characteristics to the results of field observations.

Subsequently we are planning to extend the above-proposed technique of supercomputer simulations to models with curvilinear free surface.

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B. M. Glinskyĭ; V. N. Martynov; A. F. Sapetina
Institute of Computational Mathematics and Mathematical Geophysics
Novosibirsk, Russia
vnm@nmsf.sscc.ru

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ANALYTICAL MODELING OF SUPERLONG–DISTANCE WAVE FIELDS IN THE MEDIA WITH COMPOSITE SUBSURFACE GEOMETRIES

A. G. Fatyanov

Abstract. We propose an analytical method of modeling seismic wave fields for a wide range of geophysical media: elastic, non-elastic, anisotropic, anisotropic-non-elastic, porous, random-inhomogeneous, etc. for super-remote (far) distances. As finite difference approximations are not used, there is no grid, no dispersion when computing wave fields for arbitrary media models and observation points. The analytical solution representation in the spectral domain makes possible to carry out analysis of a wave field in parts, specifically, to obtain the primary waves. Based on the developed program of computing wave fields, we carried out the simulation of seismic "ringing," on the Moon and compare it with the ray method.

Keywords: mathematical modeling, analytical solution, full wave field, primary waves, elastic, non-elastic, anisotropic-non-elastic, porous, random-inhomogeneous media; seismic ray method

Introduction

Mathematical modeling nowadays is one of the main tools for studying seismic wave propagation for various models of media. Continual improvement of measuring equipment leads, on the one hand, to higher accuracy of experimental data, and on the other, to increasing spatial and temporal scales. This makes it necessary to develop the new methods and refine the available methods for calculating wave fields. This article develops an analytical modeling method enabling us to calculate wave fields for considerable spatial and temporal scales arising in experimental work.

The possibility of analyzing the total wave field in parts is important, and often crucial, for the problems of experimental data interpretation. This article develops an analytical algorithm for modeling wave fields at long distances without bounds on the accuracy, models of media, and observation bases, and enabling us to calculate the dynamics of separate waves (primary waves and others) avoiding the restrictions of the ray method. As a result of comparison with the ray method, we show that in addition to the known geometric restrictions of the ray method there are also restrictions related to the input pulse duration.

The analytical modeling method enables us to consider various applied problems. In this article we carry out mathematical modeling of seismic wave field for an elastic Moon model, in which there is a surface zone of small velocities in the

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case of considerable spatial and temporal scales (hour-long records of experimental data). The modeling goes in the framework of a 3D planar stratified medium. Lunar seismograms differ substantially from the seismograms obtained on the Earth. The most characteristic feature of lunar seismograms is the large duration of signal exceeding hour-long records. The results of modeling show that, in the presence of a zone of small velocities in the medium, seismic "ringing" arises, leading to considerable increase in the recorded duration of seismic signal. In the first approximation we can explain the duration of seismic ringing on the Moon by resonance phenomena that arise in the wave field in the presence of a thin low-velocity stratum (regolith).

We develop an analytical modeling method for media of composite structure, including elastic, inelastic, anisotropic, anisotropic-inelastic, porous, randomly inhomogeneous, and others. The analysis of experimental data in the field near the Shugo volcano (Krasnodar region of Russia) elucidates the property of wave field related to the appearance of a series of resonances in the low-frequency range. Furthermore, the resonance is steadily shifting into the range of lower frequencies as registration distance increases, which lacks an intuitive physical explanation. This phenomenon is explained by multi-scale effects (lumpiness) in the framework of randomly inhomogeneous medium.

The analytical approach enables us to estimate the accuracy of finite difference methods. The comparison of analytical and mesh approaches yields a number of conclusions on the choice of meshsizes in the difference scheme. We find that in the mesh methods, in order to make artifact-free calculations, we should not simply take a small meshsize of the difference scheme, but it must decrease inversely proportionally to the increase in the spatial and temporal scales.

1. Statement of the Modeling Problem

The mathematical formulation of the problem of modeling wave fields in Cartesian coordinates is as follows: Determine the displacement vector components for inelastic anisotropic medium satisfying the system of equations

$$\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} &= \rho \frac{\partial^2 u_x}{\partial t^2} + f_x f(t), \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} &= \rho \frac{\partial^2 u_y}{\partial t^2} + f_y f(t), \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} &= \rho \frac{\partial^2 u_z}{\partial t^2} + f_z f(t)
\end{aligned} \tag{1}$$

with the initial conditions

$$u_x = \frac{\partial u_x}{\partial t} = u_y = \frac{\partial u_y}{\partial t} = u_z = \frac{\partial u_z}{\partial t} = 0$$

at $t = 0$ and boundary data

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0 \tag{2}$$

at $z = 0$.

Assume that the relations between the stress and deformation tensor components (as well as of the latter to the displacement components) are known. According to Volterra's principle, the anisotropic coefficients c_{ij} are replaced by the integral operators C_{ij} accounting for the influence of the elastic aftereffect:

$$C_{ij} x \equiv c_{ij} x(t) - c_{ij}^1 \int\limits_{-\infty}^{t} h_{ij}(t-\tau) x(\tau)\, d\tau, \tag{3}$$

where c^1_{ij} determine the anisotropic absorption level. The aftereffect functions (kernels) $h_{ij}(\xi)$ determine the spectral composition of absorption. Assume that the medium $\left(c_{ij}, c^1_{ij}, \rho\right)$ is piecewise constant with respect to the depth coordinate z. For a full description of anisotropic absorption, some additional physical parameters of absorption are introduced (the absorption decrements of quasi-longitudinal and quasi-transverse waves) which are determined by c^1_{ij} [1]. The components f_x, f_y, and f_z of the force describe concentrated and distributed sources of various types.

2. Analytical Modeling of Seismic Waves

For clarity, we present the analytical solution in the case of propagating SH-waves when the source lies on the free surface. Confine the discussion to considering the already classical case of transversally isotropic medium, when the symmetry axis coincides with the z-axis. In this case the problem of determining the displacement vector in cylindrical coordinates reduces to finding the only nonzero component of the displacement vector:

$$C_{66}\left(\frac{\partial^2 u_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial u_\varphi}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u_\varphi}{\partial r^2} + \frac{\partial^2 u_\varphi}{\partial z^2}\right) = \rho\frac{\partial^2 u_\varphi}{\partial t^2}, \tag{4}$$

$$C_{66}\frac{\partial u_\varphi}{\partial z}\bigg|_{z=0} = \frac{1}{2\pi}\frac{d}{dr}\frac{\delta(r)}{r}f(t), \tag{5}$$

$$u_\varphi|_{t=0} = \frac{\partial u_\varphi}{\partial t}\bigg|_{t=0} = 0. \tag{6}$$

We impose the well-known conjugation conditions on the discontinuity boundaries of the parameters.

To construct a solution, we apply the Fourier–Bessel transform with respect to the variables (r, t), while in the spectral domain (k, ω) we obtain the following two-parameter family of boundary problems in each stratum:

$$\frac{d^2 w}{dz^2} = \nu^2 w, \quad \frac{dw}{dz}\bigg|_{z=0} = f(\omega), \tag{7}$$

where $\nu^2 = k^2 - \omega^2\rho/\bigl(c_{66} - c^1_{66}\int_0^T h_{66}(t)e^{-i\omega t}\,dt\bigr)$.

Then we introduce the two unknown functions $x(z, k, \omega)$ and $y(z, k, \omega)$ such that

$$\frac{dx}{dz} = \alpha(z)x, \quad \frac{dy}{dz} = -\beta(z)y, \tag{8}$$

while x and y satisfy (7). Then we find α and β from the Riccati equations

$$\frac{d\alpha}{dz} + \alpha^2 = \nu^2, \quad \frac{d\beta}{dz} - \beta^2 = -\nu^2. \tag{9}$$

Suppose that the medium consists of N strata. Then in the halfspace we have

$$\alpha = \beta|_{z>h_N} = \nu_{N+1}. \tag{10}$$

It is not difficult to verify that the functions x and y are linearly independent [2], and so $w = x + y$. By the conditions of causality of absorption and boundedness of the solution, we have $x \equiv 0$ and $\mathrm{Re}(\nu) > 0$.

Therefore, the problem reduces to the following: Find y satisfying

$$\frac{dy}{dz} = -\beta y, \quad \frac{d\beta}{dz} - \beta^2 = -\nu^2, \quad y|_{z=0} = -\frac{f}{\beta(0)}, \tag{11}$$

$$[\beta]|_{z=h_i} = 0, \quad \beta|_{z\ge h_N} = \nu_{N+1}. \tag{12}$$

The Riccati equation admits the analytical solution in each stratum:

$$\beta(z) = \nu_i \frac{\beta_i + \nu_i \operatorname{th}(\nu_i(h_i - z))}{\nu_i + \beta_i \operatorname{th}(\nu_i(h_i - z))}. \tag{13}$$

As the spatial and temporal scales increase, arithmetic overflow occurs in the calculation of the hyperbolic tangent in (13). Thus, we can express (13) in equivalent form

$$\beta(z) = \nu_i \frac{\beta_i + \nu_i - (\nu_i - \beta_i)\exp(-2\nu_i(h_i - z))}{\beta_i + \nu_i + (\nu_i - \beta_i)\exp(-2\nu_i(h_i - z))}, \tag{14}$$

where β_i is the value of $\beta(z)$ on the upper boundary of stratum i. Finally, the original boundary problem (7) reduces to two Cauchy problems, one of which is nonlinear but not hard because it admits the analytical solution (13).

To summarize, the algorithm is as follows: Firstly, we recalculate the value of β from the halfspace to the upper boundary of stratum N (in this case they coincide). Then from (14) we find the value of β on the lower boundary of stratum $N-1$ and recalculate it on the upper boundary of $N-1$. Repeating this process N times, we evaluate $\beta(0)$. From (11) we determine y for $z = 0$ and so w as well, thus solving (7). Knowing $\beta(z)$, we find w analytically at each point of the stratified medium. Applying the inverse Fourier–Bessel transform, we determine the solution to (4)–(6) for arbitrary r, z, and t.

Numerous experimental observations indicate that we encounter various types of absorption and anisotropy. Four thin-stratum models of transversally isotropic medium of various types are suggested in [3]. Fig. 1 depicts seismograms of the vertical component and the indicatrix of the quasi-longitudinal and quasi-transverse ray velocity for a transversally isotropic medium for a source of "pressure center type" with the parameters of [3]. The wave fields are output around circles with the meshsize of 4.5° for an elastic (on the left) and an inelastic medium. We choose equal anisotropic absorption decrements $\Delta_{p\perp}, \Delta_{p\|}$ and $\Delta_{s\perp}, \Delta_{s\|}$. It is clear that each branch of the ray indicatrix in the elastic and inelastic cases corresponds to a certain wave in the total field. Furthermore, as simulations show, the opening angle of the "loop" in the elastic wave field of quasi-transverse waves is 8°–10° greater than the opening angle of the loop on the ray indicatrix of the corresponding waves. Thus, this phenomenon is of "nonray" class.

Simulations show that the introduction of absorption does not lead to significant decrease in the opening angle of the "loop" when the "nonray" domain generally decreases. Noticeable change of spectra occurs in the inelastic case. The shape of seismic pulse also changes substantially: it integrates itself. As in the "simply" inelastic medium, substantial redistribution of energy occurs and the spectral compositions of quasi-longitudinal and quasi-transverse waves change. In general, the influence of absorption depends on the decrement, the frequency, and the distance of wave propagation. For low frequencies and short distances the loss due to diversion exceeds the loss due to absorption. As the frequency and distance increase, the loss due to absorption grows and becomes dominating. This applies to both isotropic and anisotropic media. Ignoring this, we may arrive, for instance, at the incorrect determination of kinematics.

The wave portrait for four anisotropy types of [3] is studied in detail in [1]. The substantial difference of the wave portrait in anisotropic inelastic media from anisotropic elastic media is the presence of anomalous dispersion, which leads to the increase of observable periods with registration time.

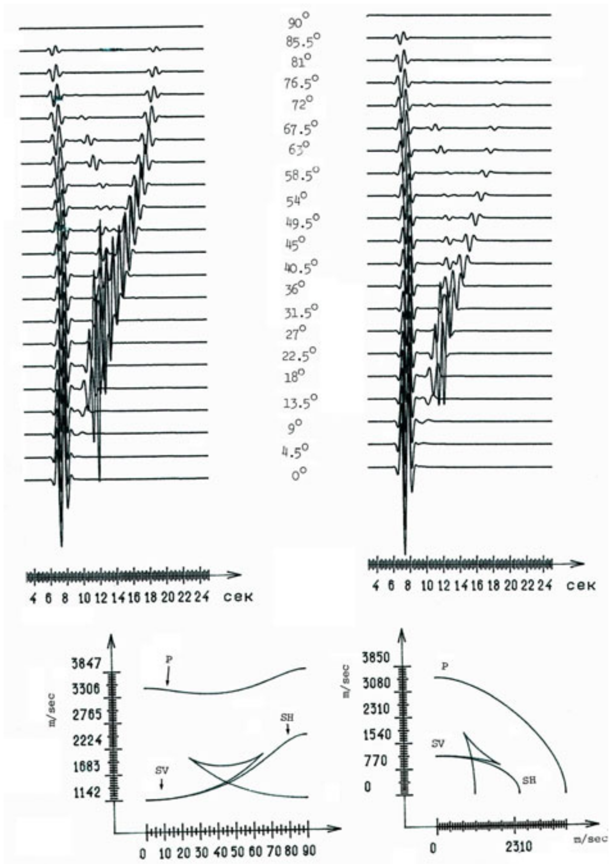

Fig. 1. The vertical component u_z and polarization diagrams for anisotropic elastic (on the left) and inelastic media

The analytical modeling method we develop enables us to consider various applied problems. With its capabilities we can model a seismic wave field for an elastic Moon model, in which there is a surface zone of small velocities (regolith) in the case of considerable spatial and temporal scales (hour-long records of experimental data). We made simulations in the framework of a 3D planar stratified medium.

Since the regolith stratum is very thin, we paid particular attention to the accuracy of calculations. For instance, in the matrix method and its modifications, the calculation of wave fields for high frequencies (thin strata) incurs accuracy loss [4]. With mesh methods, it is also practically impossible to calculate wave fields without artifacts when a thin regolith stratum is present. Modeling wave fields in this case, especially for long distances, requires the use of a small spatial meshsize, which leads to colossal computational work.

Basing on the method of [5, 6], we wrote software enabling us to model wave fields for considerable spatial and temporal scales typical of lunar experimental data. The analytical modeling method applied in this article was developed already in the 1990s. Since computers had been insufficiently powerful, simulations ran for the spatial and temporal scales on the order of 50 wavelengths. Here we do calculations for tens of thousands of wavelengths. This required some modernization of the method. To make calculations for considerable spatial and temporal scales possible on common computers, we modify the algorithm, trying to exclude large intermediate arrays, which are calculated analytically during calculation. We also introduce higher calculation accuracy, carry out additional testing, and so on.

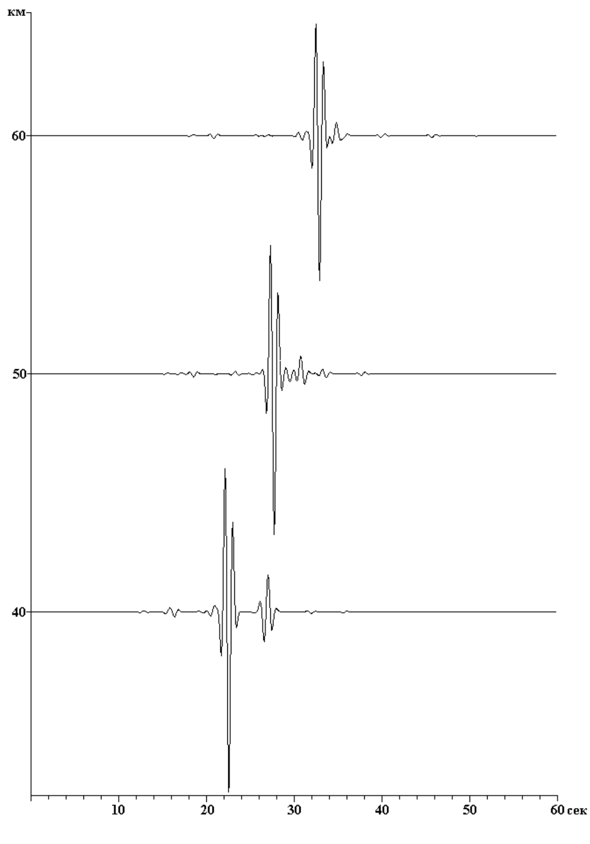

Fig. 2. Calculation of the minute-long record of the component u_z for the Moon model without the zone of small velocities (regolith). Receivers are at the distances of 40, 50, and 60 km away from the source

Finally, the analytical approach (avoiding meshes) enables us to simulate seismic ringing on the Moon on ordinary computers without using high-performance computing technology.

The scheme of simulation relies on the available Moon models [7, 8]. The first of the models used is the model of the surface part of the Moon in which we distinguish several small strata with low velocities. The upper stratum in the first model is 10 m thick regolith. The longitudinal wave velocity in regolith equals 100 m/s, the transverse wave velocity is 40 m/s [7]. The second model is a depth model reaching the center of the Moon and consisting of thickness of many kilometer strata [8]. The Moon model used in the simulations combines these models by replacing the first stratum of the depth model [8] with the strata of the model [7]. Below we present the results of simulation for a source of normal force type lying on the daylight surface with the dominating frequency of the input signal 1 Hz. In the first approximation, this source corresponds to a meteorite strike.

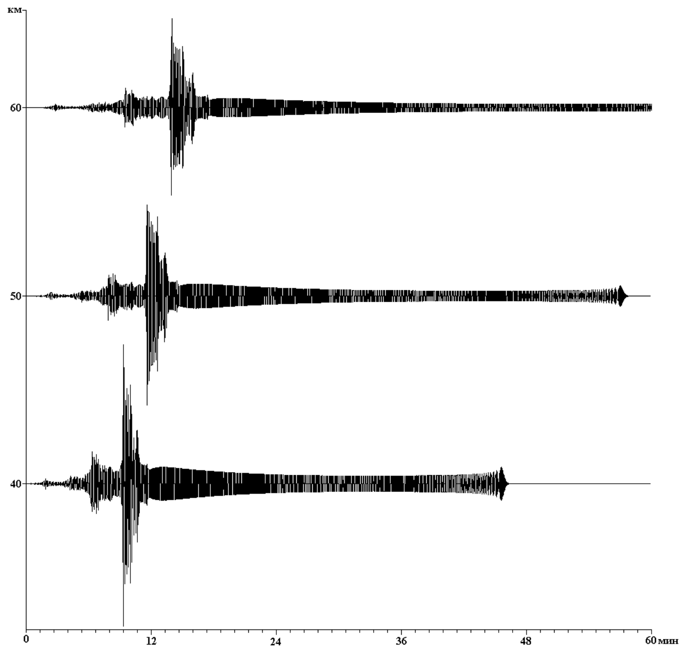

Fig. 3. Calculation of hour-long record of the component u_z
for the Moon model with a zone of small velocities (regolith).
Receivers are at the distance of 40, 50, and 60 km from the source

Lunar seismograms differ greatly from the seismograms on the Earth. The most characteristic feature of lunar seismograms is the significant duration of the seismic signal exceeding hour-long records [7], which is explained in [9]. It is assumed that seismic "ringing" is due to the high degree of inhomogeneity of the medium leading to intensive scattering with very low seismic energy absorption in the surface stratum. However, simulations of a wave field in a scattering medium of this kind have never been made.

Fig. 2 implies that in the model without regolith, the oscillations die out completely during one minute, and there is no ringing in this case. Fig. 3 presents an example of calculation for the Moon model with a zone of small velocities (regolith). It is clear that ringing in the presence of regolith is of significant duration, which at long distances exceeds an hour.

The simulations show the essential dependence of the duration of ringing on the presence of a thin low-velocity stratum consisting of regolith. We also observe significant duration of ringing in the presence of thin regolith stratum both in the model of the upper part of depth cut of the Moon [7], and in its general model [8]. However, if we take the Moon model without thin regolith stratum (Fig. 2) then wave field has duration of less than a minute, and ringing is not observed. The simulation allows us to draw the following conclusion: In the first approximation, seismic ringing on the Moon can be explained by the resonant properties of a thin stratum without invoking the scattering effects due to the high degree of inhomogeneity of the medium.

3. Calculation of Primary Waves Without Using Reflection Coefficients and Comparison with the Ray Method

There are many methods for calculating wave fields in stratified media. They are all applicable to specific domains and can calculate only total wave field without selecting separate wave types from it. At the same time, the algorithms enabling us to calculate the dynamics of separate waves are important and often crucial in the problems of modeling wave fields in composite media. In stratified inhomogeneous media with many strata the asymptotic ray method is the unique method for simulating wave field in parts. Its applicability, however, is restricted. Below we consider an algorithm for calculating primary and monotype waves for stratified inhomogeneous media with an arbitrary number of strata basing on special expansions of exact solutions, which is free of the restrictions of the ray method.

For clarity, consider the wave equation for P waves. In this case the solution in the spectral domain upon transformations is obtained on the free surface in explicit form

$$u(k,\omega) = -\frac{F(\omega)}{\nu_1}\big[1 - 2p_1 e^{-2\nu_1 h} - 2p_2 e^{-2(\nu_2+\nu_1)h}\big(1-p_1^2\big) - \cdots\big], \tag{15}$$

where $p_i = (\nu_{i+1} - \nu_i)/(\nu_{i+1} + \nu_i)$.

Let us elucidate the physical meaning of the parameters in (15). For greater clarity, consider the case of planar waves. In this case $p_i = (c_i - c_{i+1})/(c_i + c_{i+1})$. Therefore, (15) is just the spherical analog of the well-known planar representation of singly reflected waves [10].

An algorithm for calculating primary longitudinal and transverse waves for elastic and inelastic media is developed in [11]. For comparison between the analytical and ray methods, see [12].

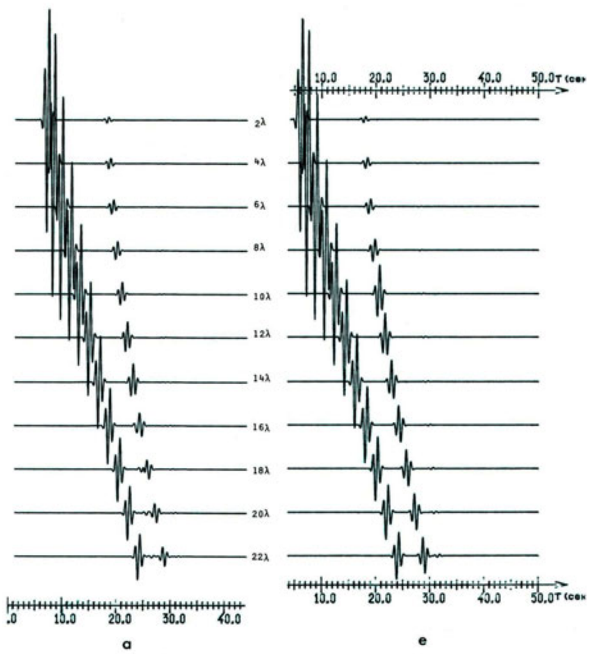

Fig. 4. Comparison of the analytical (a) and ray (e) methods for PP waves

Fig. 4 depicts an example that compares analytical and ray methods for primary longitudinal (PP) waves for a two-stratum model of a medium at the same scale. The normalization is with respect to the direct wave. Differences in the amplitude and phase are visible. Furthermore, the differences are present not only at the exit point of the head wave, as previously presumed. We also observe differences in some domain depending on the input pulse duration. The results of [12] show that in addition to the geometric restrictions of the ray method there are restrictions related to the duration of the input pulse. The greater pulse duration, the greater the domain in which the ray method is inaccurate.

4. Analytical Modeling of Wave Fields in Media of Composite Structure

Often the structure of an inhomogeneous medium is such that its characteristic sizes in depth, for instance, vary in a wide range. It is known [13] that in this case it is practically impossible to consider this problem in the framework of deterministic approach. In any case, deterministic approach requires considerably more complicated calculations. To construct a model of multi-scale inhomogeneity, we use the telegraph random process $\xi(z) = a(-1)^{n(0,z)}$ of [14], where $n(z_1, z_2)$ is a random sequence of integers describing the number of jumps on the interval (z_1, z_2). The probability of n points on (z_1, z_2) is given by Poisson's formula

$$\mathrm{P}_{n(z_1,z_2)=n} = \frac{\langle n(z_1, z_2)\rangle^n}{n!} e^{-\langle n(z_1,z_2)\rangle},$$

where $\langle n(z_1, z_2)\rangle = \nu|z_2 - z_1|$ is the averaged value of the points on (z_1, z_2), while ν is the averaged number of points per unit distance. The length of the interval between adjacent jumps is distributed exponentially.

We illustrate the construction of equations for a multi-scale medium on the example of the wave equation for longitudinal waves in cylindrical coordinates:

$$\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \tag{16}$$

The Fourier–Bessel transform converts (16) into (omitting inessential indices)

$$\frac{d^2 u}{dz^2} = k^2 u - \frac{\omega^2}{c^2} u = \nu_p^2 u. \tag{17}$$

In order to consider a multi-scale medium in (17), use the telegraph random process $\xi(z)$:

$$\frac{d^2 u}{dz^2} = k^2 u - \frac{\omega^2}{c^2}[1 + \xi(z)]u \tag{18}$$

The solution to (18) as some functional of $\xi(z)$ is a random quantity. With practical applications in mind, consider the mean value $\langle u\rangle$ of the fluctuating field. The exact complete equation

$$\begin{gathered}\frac{d^4\langle u\rangle}{dz^4} + 4\nu\frac{d^3\langle u\rangle}{dz^3} + \left(4\nu^2 + \frac{\omega^2}{c^2} - \nu_p^2\right)\frac{d^2\langle u\rangle}{dz^2} \\ -4\nu\nu_p^2\frac{d\langle u\rangle}{dz} - \left(\frac{\omega^4}{c^4}a^4 + 4\nu^2\nu_p^2 + \frac{\omega^2}{c^2}\nu_p^2\right)\langle u\rangle = 0\end{gathered} \tag{19}$$

for the mean of the field is obtained in [15].

Therefore, instead of the ordinary second-order wave equation (17), in the case of a randomly inhomogeneous medium we obtain a fourth-order equation for the mean value of the field. Complement (19) with appropriate boundary conditions for the mean value of the field [16]. The parameter a in (19) determines the magnitude of multi-scale inhomogeneity, while $r_0 = 1/2\nu$ is the characteristic size of inhomogeneity (lumpiness) of the medium. The parameters a and r_0 can be arbitrary. An analytical method for calculating multi-scale waves appears in [16].

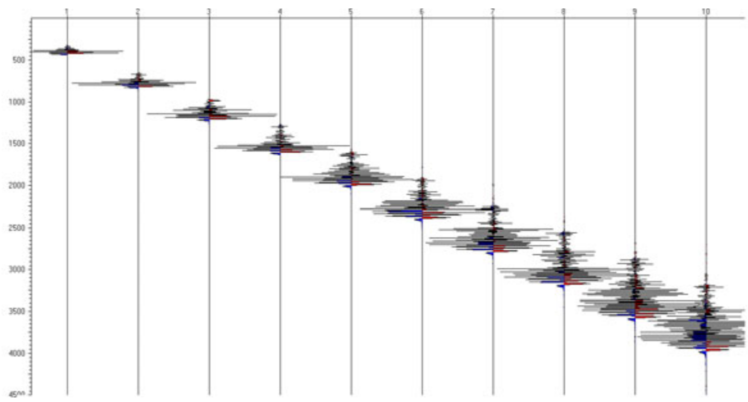

Fig. 5. Wave field on the daylight surface for a multi-scale inhomogeneous medium. Characteristic size of inhomogeneity $\lambda/16$

Fig. 5 depicts wave fields for long distances in the source/receiver system with large multiple scales of inhomogeneity. The output is on the free surface. The characteristic size of inhomogeneity is $r_0 = \lambda/16$, where λ is the wavelength. The first receiver lies at the distance of 30λ from the source and the last, at 300λ. It is clear from Fig. 5 that at short distances the effects of multiple scales are considerably smaller than at long distances. As the receiver moves away from the source of waves, the picture changes substantially. Energy is considerably redistributed into the domain of low frequencies.

Inspecting the experimental data in the field near the Shugo volcano (the Krasnodar region of Russia), we came across a feature of the wave field related to the appearance of a series resonances in the low-frequency range. Furthermore, the resonance shifts monotonely to lower frequencies as the distance registration increases [17], which lacks an intuitive physical explanation. This phenomenon can be simply explained by the multi-scale effects (lumpiness) of the medium. It is clear from Fig. 5 that the frequency of a multi-scale wave decreases monotonely as distance grows. This fully explains the effect of monotone decrease in the frequency with the growth of distance detected in practice.

The analytical method for modeling is developed for a wide range of geophysics media, including elastic, inelastic, anisotropic, anisotropic-inelastic, porous, randomly inhomogeneous and so on. The sources in the descriptions of seismic, seismological, and vibrational processes can be concentrated and distributed [18]. Since the solution is given by analytical expressions, we can do calculations at long distances on ordinary computers without the need of high-performance computing technology.

Moreover, the analytical method can serve for numerical control of the accuracy of finite-difference methods. Comparison between analytical and mesh methods is made in [19]. It turns out that in mesh methods, to make artifact-free calculations, we should take not just a small meshsize of the difference scheme, but it must also decrease inversely proportionally to the increase of the record duration (growth of the spatial and temporal scales).

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Novosibirsk

A. G. Fatyanov
Institute of Computational Mathematics and Mathematical Geophysics
Novosibirsk, Russia
`fat@nmsf.sscc.ru`